Solutions to MATH256-103-2019, HW#2

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1. Abel's Formula $W = ce^{-\int p(t)\,dt}$

(a) $p = x$, $W = c\, e^{-\int x\,dx} = ce^{-\frac{x^2}{2}}$

(b) $p = -\frac{2x}{x^2}$, $W = ce^{\int \frac{2}{x}dx} = cx^2$

(c) $p = \frac{2\cos t}{\sin t}$, $W = ce^{-\int \frac{2\cos t}{\sin t}dt} = ce^{-2\ln|\sin t|} = \frac{c}{\sin^2 t}$ (10pts)

(d) $p = \frac{2x}{1+x^2}$, $W = c\,e^{-\int \frac{2x}{1+x^2}dx} = ce^{-\frac{1}{2}\ln(1+x^2)} = \frac{c}{\sqrt{1+x^2}}$

2. (a) $r^2+3r-4=0$, $r_1=-4$, $r_2=1$, $y = c_1e^{-4t}+c_2e^{t}$

(b) $2r^2+3r+1=(2r+1)(r+1)=0$, $r_1=-\frac{1}{2}$, $r_2=-1$, $y=c_1e^{-\frac{1}{2}t}+c_2e^{-t}$

(c) $r^2+2r+5=0 \Rightarrow r=-1\pm 2i$, $\lambda=-1$, $\mu=2$, $y=c_1e^{-t}\cos 2t+c_2e^{-t}\sin(2t)$ (10pts)

(d) $r^2+9=0 \Rightarrow r=\pm 3i$, $\lambda=0$, $\mu=3$, $y=c_1\cos 3t+c_2\sin 3t$

(e) $r^2+6r+9=0 \Rightarrow r_1=r_2=-3$, $y=c_1e^{-3t}+c_2te^{-3t}$

3. (a) $y=c_1e^{-t}\cos\sqrt{2}t+c_2e^{-t}\sin\sqrt{2}t$

$$\left.\begin{array}{l} y(0)=1 \Rightarrow c_1 = -1 \\ y'(0)=2 \Rightarrow -c_1+\sqrt{2}c_2=2 \end{array}\right\} \Rightarrow c_1=-1,\ c_2=\frac{1}{\sqrt{2}}$$ (20pts)

(b) $y=c_1e^{-t}+c_2e^{-3t}$, $y(-1)=2 \Rightarrow c_1e+c_2e^3=2$

$$\left.\begin{array}{l} c_1e+c_2e^3=2 \\ y'(-1)=1 \Rightarrow -c_1e-3e^3c_2=1 \end{array}\right\}\ c_2=-\frac{3}{2}e^{-3},\ c_1=\frac{7}{2}e^{-1}$$

(c) $y=c_1e^{-\frac{2}{3}t}+c_2te^{-\frac{2}{3}t}$, $y(0)=2 \Rightarrow c_1=2$, $y'(0)=-1 \Rightarrow -\frac{2}{3}c_1+c_2=-1$

$\Rightarrow c_2=\frac{1}{3}$

(d) $y=c_1+c_2e^{\frac{5}{2}t}$, $y(1)=1 \Rightarrow c_1+e^{\frac{5}{2}}c_2=0$, $y'(1)=-1 \Rightarrow \frac{5}{2}c_2e^{\frac{5}{2}}=-1$

$\Rightarrow c_2=-\frac{2}{5}e^{-\frac{5}{2}}$, $c_1=\frac{2}{5}$

4. $y_2=y_1\int\frac{W}{y_1^2}$ (20pts)

(a) $W=ce^{-\int\frac{3t}{t^2}dt}=ct^{-3}$, $y_2=t^{-1}\int\frac{ct^{-3}}{t^{-2}}=t^{-1}\ln t$

(b) $W=ce^{\int\frac{t+2}{t}dt}=ct^2e^t$, $y_2=t\int\frac{ct^2e^t}{t^2}=cte^t$

(c) $W=c\,e^{\int\frac{x}{x-1}dx}=ce^x(x-1)$, $y_2=e^x\int\frac{e^x(x-1)}{e^{2x}}=e^x\int e^{-x}(x-1)=e^x\cdot(xe^{-x})=x$

$\therefore\ W = ce^{-\int \frac{1}{x}dx} = \frac{c}{x}$, $\quad y_2 = x^{-\frac{1}{2}}\sin x \int \frac{\frac{c}{x}}{x^{-1}\sin^2 x} = x^{-\frac{1}{2}}\sin x \int \frac{1}{\sin^2 x}$

$= -x^{-\frac{1}{2}}\sin x \cdot \left(\frac{\cos x}{\sin x}\right) = -x^{-\frac{1}{2}}\cos x$

5. (a) $r(r-1)+3r-2=0 \Rightarrow r^2+2r-2=0, \Rightarrow r = \cancel{\frac{-4\pm\sqrt{4^2+8}}{2} = 2\pm\sqrt{8}}\ -1\pm\sqrt{3}$

$y = c_1 t^{-1+\sqrt{3}} + c_2 t^{-1-\sqrt{3}}$

(b) $2r(r-1)-4r+1=0 \Rightarrow 2r^2-6r+1=0 \Rightarrow r = \frac{6\pm\sqrt{36-8}}{2\times 2} = \frac{3\pm\sqrt{7}}{2}$

$y = c_1 t^{\frac{3+\sqrt{7}}{2}} + c_2 t^{\frac{3-\sqrt{7}}{2}}$

20 pts

(c) $r(r-1)+3r+1=0 \Rightarrow r^2+2r+1=0 \Rightarrow r_1=r_2=1$

$y = c_1 t^{-1} + c_2 t^{-1}\ln t$

(d) $r(r-1)+5r+13=0 \Rightarrow r^2+4r+13=0 \Rightarrow r=-2\pm 3i$

$y_1 = c_1 t^{-2}\cos(3\ln t) + c_2 t^{-2}\sin(3\ln t)$

6. (a) $y_1 = e^{-t}\cos t,\ y_2 = e^{-t}\sin t$

$y_p = A\cos t + B\sin t \Rightarrow y_p' = -A\sin t + B\cos t,\ y_p'' = -A\cos t - B\sin t$

$y_p''+2y_p'+2y_p = (-A+2B+2A)\cos t + (-B-2A+2B)\sin t = \sin t$

$\left.\begin{matrix} A+2B=0 \\ B-2A=1 \end{matrix}\right\} \Rightarrow \begin{matrix} A=-\frac{2}{5} \\ B=\frac{1}{5} \end{matrix}$

40 pts

$y = -\frac{2}{5}\cos t + \frac{1}{5}\sin t + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

(b) $y_1 = e^{-t}\cos t,\ y_2 = e^{-t}\sin t,$

$y_p = At^2+Bt+C \Rightarrow y_p' = 2At+B,\ y_p''=2A$

$2A + 2(2At+B) + 2(At^2+Bt+C) = 3t^2 \Rightarrow$
$2A=3 \Rightarrow A=\frac{3}{2}$
$4A+2B=0 \Rightarrow B=-3$
$2A+2B+2C=1 \Rightarrow C=2$

$y = \frac{3}{2}t^2-3t+2 + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

(c) $y_p = (A\cos 2t + B\sin 2t)e^{-t}$, $y_p' = (-2A\sin 2t + 2B\cos 2t - A\cos 2t - B\sin 2t)e^{-t}$

$= (-(2A+B)\sin 2t + (2B-A)\cos 2t)e^{-t}$

$y_p'' = (-2(2A+B)\cos 2t - 2(2B-A)\sin 2t + (2A+B)\sin 2t - (2B-A)\cos 2t)e^{-t}$

$= ((-3A-4B)\cos 2t + (4A-3B)\sin 2t)e^{-t}$

$y_p''+2y_p'+2y_p = e^{-t}\sin 2t \Rightarrow$
$-3A-4B+2(2B-A)+2A=0$
$(4A-3B)-2(2A+B)+2B=1$

$-3A = 0$

$-3B = 1 \Rightarrow B = -\frac{1}{3}$

$y = -\frac{1}{3}\sin 2t\, e^{-t} + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

(d) $y_p = t(A\cos t + B\sin t)e^{-t}$

$y_p' = (A\cos t + B\sin t)e^{-t} + t[(B-A)\cos t + (-A-B)\sin t]e^{-t}$

$y_p'' = (-A\sin t + B\cos t)e^{-t} + [(B-A)\cos t + (-A-B)\sin t]e^{-t} + t[\cdot]$
$- (A\cos t + B\sin t)e^{-t}$

$y_p'' + 2y_p' + 2y_p = (-A\sin t + B\cos t + (B-A)\cos t + (-A-B)\sin t - (A\cos t + B\sin t)$
$+ 2(A\cos t + B\sin t)] e^{-t}$

$\cancel{-2A+2B=0}$, $\cancel{2A - A - B = 1}$ $+B$ $\cancel{\Rightarrow A = B = 1}$

$2A + B + B - A - A = 1 \Rightarrow B = \frac{1}{2}$

$2B - A - A - B - B = 0 \Rightarrow A = 0$

$y = t(\frac{1}{2}\cos t \cancel{+ \sin t})e^{-t} + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

(e) $y'' + 2y' + y = e^{t} + e^{-t}$, $y_1 = e^{-t}$, $y_2 = te^{-t}$

$y_p = Ae^{t} + Bt^2e^{-t} \Rightarrow \cancel{y_p'' + 2y_p' + y_p}$

$y_p' = Ae^{t} + 2Bte^{-t} - Bt^2e^{-t}$

$y_p'' = Ae^{t} \cancel{+ 4Bte^{-t}} + 2Be^{-t} - 4Bte^{-t} + Bt^2e^{-t}$

$y_p'' + 2y_p' + y_p = 4Ae^{t} + 2Be^{-t} = e^{t} + e^{-t}$

$A = \frac{1}{4}$, $B = \frac{1}{2}$

$y = \frac{1}{4}e^{t} + \frac{1}{2}t^2e^{-t} + c_1e^{-t} + c_2te^{-t}$

7. (a). $y_1 = \cos t$, $y_2 = \sin t$

$y_p = t^{s_1}[At^3 + Bt^2 + Ct + D] + t^{s_2}[Et^2 + Ft + G]e^{t} + t^{s_3}[(Ht+I)\cos t + (Jt+K)\sin t]$

$s_1 = 0$, $s_2 = 0$, $s_3 = 1$

20pts

(b) $y'' + 4y' + 3y = \frac{1}{2}e^{t} - e^{-t} + t^2 - 2 + \cos 2t$

$y_1 = e^{-t}$, $y_2 = e^{-3t}$

$y_p = t^{s_1}Ae^{t} + t^{s_2}Be^{-t} + t^{s_3}(Ct^2 + Dt + E) + t^{s_4}(F\cos 2t + G\sin 2t)$

$s_1 = 0$, $s_2 = 1$, $s_3 = 0$, $s_4 = 0$

8. We discuss 3 cases:

Case 1. $r_1 \neq r_2$, $b^2-4ac>0$

$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} < 0$

Since $r_1 r_2 = \frac{c}{a} > 0 \Rightarrow r_1 < 0$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \rightarrow 0$ as $t \rightarrow +\infty$

20 pts

Case 2: r_1, r_2 complex, $b^2-4ac<0$

$r_1, r_2 = \lambda \pm i\mu$, $\lambda = -\frac{b}{2a}$, $\mu = \frac{\sqrt{4ac-b^2}}{2a}$

$y = c_1 e^{-\frac{b}{2a}t} \cos\mu t + c_2 e^{-\frac{b}{2a}t} \sin\mu t \rightarrow 0$ as $t \rightarrow +\infty$ since $\frac{b}{2a} > 0$

Case 3: $r_1 = r_2 = -\frac{b}{2a}$, $b^2-4ac=0$

$y = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}$

$\frac{b}{2a} > 0$, $e^{-\frac{b}{2a}t} \rightarrow 0$, $t e^{-\frac{b}{2a}t} \rightarrow 0$ as $t \rightarrow +\infty$

All together, $y(t) \rightarrow 0$ as $t \rightarrow +\infty$.