(20 points) 1. For the following system of ordinary differential equations, (a) find the general solutions; (b) classify the types (saddle, node (source or sink), spirals) and the stability; (c) draw a few trajectories

$$(1)\ \mathbf{x}' = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}\mathbf{x}, \qquad (2)\ \mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x}$$

(1) $A = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$

$\det(A - \lambda I) = (\lambda+1)^2 - 4 = 0$

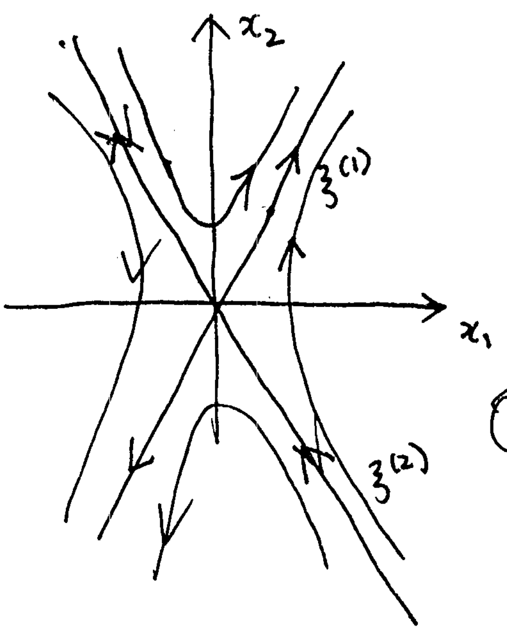
$\lambda_1 = 1, \quad \lambda_2 = -3$

For $\lambda_1 = 1$, $\xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = -3$, $\xi^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$x = c_1 e^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ +6

$\lambda_1 > 0 > \lambda_2$: saddle unstable +2



+2

(2) $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

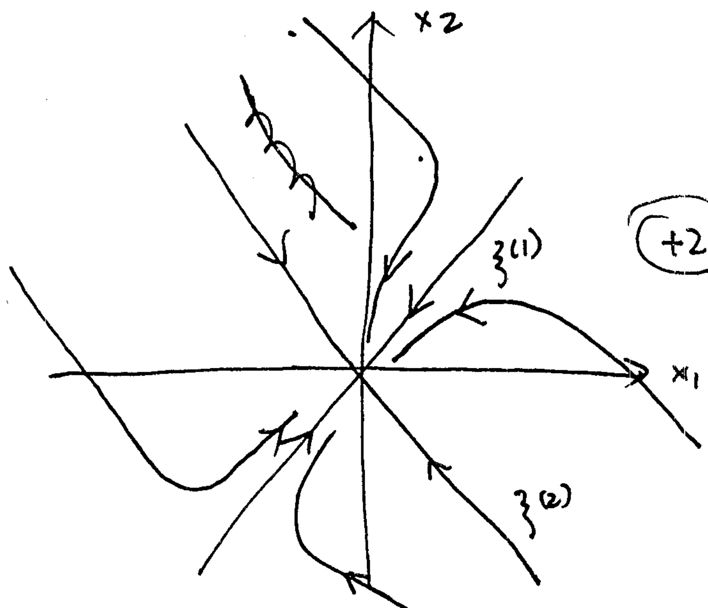
$\det(A - \lambda I) = (\lambda+2)^2 - 1 = 0$

$\lambda_1 = -1, \quad \lambda_2 = -3$

$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ +6

$\lambda_1 < 0, \lambda_2 < 0$: sink, stable +2



+2

$t \to +\infty, \quad x \approx e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$t \to -\infty, \quad x \approx e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(10 points) 2. Find the general solutions to the following system of ordinary differential equations

$$x' = \begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix} x$$

+5

$A = \begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix}$ $\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ -5 & -1-\lambda \end{pmatrix}$

$= (\lambda+1)(\lambda-3)+5 = \lambda^2-2\lambda+2=0$

$\lambda = 1 \pm i$

$\lambda_1 = 1+i$, $\begin{pmatrix} 3-(1+i) & 1 \\ -5 & -1-(1+i) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, $\xi^{(1)} = \begin{pmatrix} 1 \\ -2+i \end{pmatrix}$

+5

$e^{(1+i)t}\begin{pmatrix} 1 \\ -2+i \end{pmatrix} = \begin{pmatrix} e^t\cos t + i e^t \sin t \\ (-2+i)(e^t \cos t + i e^t \sin t) \end{pmatrix}$

$= \begin{pmatrix} e^t \cos t \\ -2e^t\cos t - e^t \sin t \end{pmatrix} + i \begin{pmatrix} e^t \sin t \\ e^t \cos t - 2e^t \sin t \end{pmatrix}$

$x = c_1 \begin{pmatrix} e^t \cos t \\ -2e^t \cos t - e^t \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^t \sin t \\ e^t \cos t - 2e^t \sin t \end{pmatrix}$

(~~25~~ 15 points) 3. Use the method of undetermined coefficients to obtain the general solutions of

$$x' = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} x + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

1) $A = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$

$\det(A - \lambda I) = (\lambda+1)(\lambda-4) + 6 = \lambda^2 - 3\lambda + 2 = 0$

$\lambda_1 = 1, \quad \lambda_2 = 2$

For $\lambda = \lambda_1$, $\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\lambda = \lambda_2$, $\xi^{(2)} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

+4

So $x_h = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

2) $x_p = \vec{a} t e^t + \vec{b} e^t$ — 3

$x_p' = \vec{a} t e^t + (\vec{a} + \vec{b}) e^t$

$= A a t e^t + A b e^t + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\vec{a} = \vec{a}$

$Ab = a + b - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | 3

+~~8~~0

4

$\vec{a} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Ab - b = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\left.\begin{array}{l} -2b_1 - 2b_2 = k - 1 \\ 3b_1 + 3b_2 = -k \end{array}\right\} \Rightarrow 3(k-1) = -2k \Rightarrow k = \frac{3}{5}$

$-2b_1 - 2b_2 = -\frac{2}{5} \Rightarrow b_1 + b_2 = \frac{1}{5}, \quad b_1 = \frac{1}{5}, \ b_2 = 0$

$x_p = \frac{3}{5} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^t + \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} e^t$

3) $x = x_h + x_p$ +1

(~~20~~ 15 points) 4. Use the method of variation of parameters to obtain the general solutions of

$$x' = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ t^{-2} \end{pmatrix}, \ t > 0.$$

1) $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$

$\det(A - \lambda I) = \lambda^2 - 4 + 4 = 0, \quad \lambda = 0$

$\lambda = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x^{(1)} = e^{0t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \eta$

$A\eta = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$2\eta_1 - \eta_2 = 1 \Rightarrow \quad \eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$x^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} t \\ 2t-1 \end{pmatrix}$

$\Psi = \begin{pmatrix} 1 & t \\ 2 & 2t-1 \end{pmatrix}$

+7

2) $x_p = \Psi \vec{c}(t) \Rightarrow \Psi c' = \begin{pmatrix} 0 \\ t^{-2} \end{pmatrix}$

$c_1' + t c_2' = 0$

$2c_1' + (2t-1)c_2' = t^{-2} \Rightarrow -c_2' = t^{-2} \Rightarrow c_2 = t^{-1}$

$c_1' = -t c_2' = +t^{-1} \Rightarrow c_1 = \ln t$

+7

So $x_p = \begin{pmatrix} 1 & t \\ 2 & 2t-1 \end{pmatrix}\begin{pmatrix} \ln t \\ t^{-1} \end{pmatrix}$

3) $x = \begin{pmatrix} 1 & t \\ 2 & 2t-1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 1 & t \\ 2 & 2t-1 \end{pmatrix}\begin{pmatrix} \ln t \\ t^{-1} \end{pmatrix}$

+1

(10 points) 5. Compute the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2\pi \\ \cos t, & 2\pi \le t < +\infty \end{cases}$$

Hint: $\cos(t) = \cos(t - 2\pi)$.

(List of Laplace transform formulas is attached at the last page)

$$f(t) = 1 + (\cos t - 1)\, u_{2\pi}(t)$$

$$= 1 + (\cos(t-2\pi) - 1)\, u_{2\pi}(t)$$

+5

$$L[f](s) = \frac{1}{s} + e^{-2\pi s}\left[\frac{s}{s^2+1} - \frac{1}{s}\right]$$

+5

$$= \frac{1}{s} - \frac{1}{s(s^2+1)} e^{-2\pi s}$$

5 points) 6. Use the method of Laplace transform to solve

$$y'' + y = g(t), \quad y(0) = 0, y'(0) = 0$$

where

$$g(t) = \begin{cases} -1, & 0 \le t < 2 \\ t-3, & 2 \le t < 3, \\ 0, & 3 \le t < +\infty \end{cases}$$

(List of Laplace transform formulas is attached at the last page)

$$g(t) = -1 + (t-2)u_2 - (t-3)u_3 \quad \text{— } 3$$

$$L[g](s) = -\frac{1}{s} + e^{-2s}\frac{1}{s^2} - e^{-3s}\frac{1}{s^2} \quad \text{— } 3$$

$$s^2 Y + Y = L[g](s)$$

$$Y(s) = -\frac{1}{s(s^2+1)} + \frac{1}{s^2(s^2+1)}\left(e^{-2s} - e^{-3s}\right) \quad \text{— } 2$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}, \quad L^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t \quad \text{- } 2$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}, \quad L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - \sin t \quad \text{- } 2$$

$$y(t) = L^{-1}[Y(s)] = -1 + \cos t + (t-2-\sin(t-2))u_2(t) - (t-3-\sin(t-3))u_3(t) \quad 3$$

(15 points) 7. Use the method of Laplace transform to solve

$$y'' - 3y' + 2y = \delta(t-2), \quad y(0) = 0, \ y'(0) = 1$$

(List of Laplace transform formulas is attached at the last page)

$$s^2 Y(s) - s\,y(0) - y'(0) - 3\,(s\,Y(s) - y(0)) + 2\,Y(s) = e^{-2s}$$

$$(s^2 - 3s + 2)\,Y(s) = 1 + e^{-2s}$$

+3

$$Y(s) = \frac{1}{s^2-3s+2} + \frac{1}{s^2-3s+2} e^{-2s}$$

$$\frac{1}{s^2-3s+2} = \frac{1}{(s-1)(s-2)} = \frac{a}{s-1} + \frac{b}{s-2} = \frac{-1}{s-1} + \frac{+1}{s-2}$$

+4

$$L^{-1}\left[\frac{1}{s^2-3s+2}\right] = -e^{t} + e^{2t}$$

So $$y(t) = L^{-1}[Y(s)] = \underbrace{-e^{t} + e^{2t}}_{2} + \underbrace{\left(-e^{t-2} + e^{2(t-2)}\right) u_2(t)}_{+4}$$