Be sure this exam has 16 pages including the cover.

The University of British Columbia

MATH 256, Section 201 and Section 202

Final Exam–April 18, 2018, 3.30pm – 6.00pm

2.5 Hours

Name ________________________ **Signature** ________________________

Student Number ________________ **Section** ________________________

This exam consists of **9** questions. No notes. Simple numerics calculators are allowed. A list of Laplace Transforms is provided on the last page. Write your answer in the blank page provided.

Problem	max score	score
1.	10	
2.	8	
3.	15	
4.	15	
5.	12	
6.	8	
7.	8	
8.	12	
9.	12	
total	100	

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

(10 points) 1. Solve the following first order equation for $y(t)$ and state the interval of existence:

$$\sin(t)y' + 2\cos(t)y = \sin^3(t)y^2, \; y\left(\frac{\pi}{2}\right) = 1.$$

Hint: let $v(t) = 1/y(t)$.

(8 points) 2. Solve the following initial value problem for $y(t)$ and state the interval of existence:

$$y' = 2y^2 + 2xy^2, \quad y(0) = 1.$$

(15 points) 3. (a) Given that $y_1(x) = \sin(x^2)$ is a solution to the following ODE, use reduction of order to find the second solution $y_2(x)$ and hence find the general solution:

$$xy'' - y' + 4x^3 y = 0.$$

(b) Use the method of variation of parameters to find the general solution of

$$xy'' - y' + 4x^3 y = 2x^3.$$

Hint: $\int [x \cot(x^2)]dx = \frac{1}{2} \ln|\sin(x^2)|$, $\int [x/\sin^2(x^2)]dx = -\frac{1}{2}\cot(x^2)$.

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(15 points) 4. A mass of $1kg$ is attached to a spring with Hooke's constant $4kg/s^2$ without any damping, so that the displacement $u(t)$ of the mass about its resting position is governed by the ODE

$$u'' + 4u = F(t)$$

and $F(t)$ is a time-dependent forcing.

(a) Find the solution with the following initial condition

$$u(0) = 1, \quad u'(0) = 2,$$

when $F(t) = 0$. Write the solution in the form $R\cos(\omega t - \delta)$.

(b) For each of the following functions $F(t)$ *write down* the form of the particular solution. (i) $F(t) = 2$, (ii) $F(t) = e^t$, (iii) $F(t) = \sin(t) + 2\cos(t) + e^{2t}$, (iv) $f(t) = \cos(2t)$. For case (iv), what phenomenon does the solution represent? If you don't know its name, describe the behavior of the solution as $t \to +\infty$.

(c) A viscous damper with strength $\gamma > 0$ is added to the system, so that the equation satisfied by the mass' displacement is given by

$$u'' + \gamma u' + 4u = F(t)$$

How large does γ need to be so that the solutions become overdamped, i.e. that the displacement ceases to oscillate when $F(t) = 0$?

(d) What is the solution when $\gamma = 5$ and $F(t) = 10\sin(2t)$ and the initial conditions are $u(0) = 1$, $u'(0) = 0$? Describe the long-time behavior as $t \to \infty$.

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(12 points) 5. Use any method to find the general solution to the following system of ordinary differential equations for $\mathbf{x}(t)$:

$$\mathbf{x}' = \begin{pmatrix} 1 & 5 \\ -2 & -5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \cos t \\ 0 \end{pmatrix}.$$

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(8 points) 6. The system

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$$

contains a constant parameter α. Describe the types and stability of the origin (i.e. stable node, unstable spiral etc.) for the ranges of the constant α:
(i) $\alpha < -2$; (ii) $-2 < \alpha < 0$; (iii) $0 < \alpha < 2$; (iv) $\alpha = 0$.

(8 points) 7. Find the Laplace transforms of the following functions, using any method (e.g. either calculating it explicitly using the formula or by using any results in the table).

(a)

$$f(t) = \begin{cases} 1 & \text{if } t < 1, \\ t & \text{if } 1 \le t < 2, \\ 1 + \sin \pi t & \text{if } t \ge 2 \end{cases} .$$

(b)

$$f(t) = \int_0^t \sin(t-\tau) e^{2\tau} \, d\tau.$$

(List of Laplace transform formulas is attached on the last page)

(12 points) 8. The Laplace transform of the solution to a differential equation is given by

$$Y(s) = \frac{s+2}{s^2+2s+2} + \frac{2e^{-s}}{s(s^2+2s+2)} + \frac{e^{-2s}}{s^2+2s+2}.$$

(a) Find the solution $y(t)$ by inverting $Y(s)$. Hint:

$$\frac{2}{s(s^2+2s+2)} = \frac{1}{s} - \frac{s+1}{s^2+2s+2} - \frac{1}{s^2+2s+2}$$

(b) Provide a second-order differential equation for $y(t)$ and initial values for $y(0)$ and $y'(0)$ that has a transformed solution given by $Y(s)$.

(List of Laplace transform formulas is attached at the last page)

(12 points) 9. Use the method of separation of variables to solve the following wave equation

$$u_{tt} = 4u_{xx}, \quad 0 < x < \pi, \quad t > 0;$$

$$u(0,t) = 0, \quad u(\pi,t) = \pi;$$

$$u(x,0) = \begin{cases} 0, & 0 < x < \frac{\pi}{2} \\ x, & \frac{\pi}{2} \le x < \pi \end{cases}, \quad u_t(x,0) = \sin(3x).$$

Hint: You first need to find a steady state solution.

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List of Laplace transform formulas:

$f(t)$	$\mathcal{L}[f](s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$e^{\lambda t}\cos(\mu t)$	$\frac{s-\lambda}{(s-\lambda)^2+\mu^2}$
$e^{\lambda t}\sin(\mu t)$	$\frac{\mu}{(s-\lambda)^2+\mu^2}$
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$
$f(t-c)u_c(t)$	$e^{-cs}\mathcal{L}[f](s)$
$f(t-c)H(t-c)$	$e^{-cs}\mathcal{L}[f](s)$
$\delta(t-c)$	e^{-cs}
$\int_0^t f(t-\tau)g(\tau)d\tau$	$\mathcal{L}[f](s)\mathcal{L}[g](s)$
$f^{'}(t)$	$s\mathcal{L}[f](s)-f(0)$
$f^{''}(t)$	$s^2\mathcal{L}[f](s)-sf(0)-f^{'}(0)$

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st}f(t)dt$$

$$u_c(t) = H(t-c) = \begin{cases} 0, t < c \\ 1, t \geq c \end{cases}$$