Systems of ODEs: inhomogeneous Case

In the following, I will illustrate the three methods by the same example

Ex. 1. Use the Method of Undetermined Coefficients to solve

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

Sol'n: $x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$, $x^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

$$x_p = t\vec{a}e^{-t} + \vec{b}e^{-t}, \quad \text{where } \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$x_p' = \vec{a}e^{-t} - t\vec{a}e^{-t} - \vec{b}e^{-t}$$

$$= -t\vec{a}e^{-t} + (\vec{a} - \vec{b})e^{-t}$$

$$Px_p + g = tP\vec{a}e^{-t} + P\vec{b}e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

$$x_p' = Px_p + g \Rightarrow$$

$$-t\vec{a}e^{-t} + (\vec{a} - \vec{b})e^{-t} = tP\vec{a}e^{-t} + P\vec{b}e^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

Equal te^{-t}-term: $\quad -\vec{a} = P\vec{a}$ — (1)

Equal e^{-t}-term: $\quad \vec{a} - \vec{b} = P\vec{b} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ — (2)

From (1) $\Rightarrow$ $\vec{a} = k\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (since $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector corresponding to $r_2 = -1$)

Substituting into (2) $\Rightarrow$

$$k\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$

$$k - b_1 = b_1 + b_2 \quad \text{— (3)}$$

$$-2k - b_2 = 4b_1 + b_2 + 1 \quad \text{— (4)}$$

$$2b_1 + b_2 = k$$

$$4b_1 + 2b_2 = -2k-1 \quad \Rightarrow \quad 2b_1 + b_2 = -k - \tfrac{1}{2}$$

Hence $k = -k - \frac{1}{2} \Rightarrow k = -\frac{1}{4}$

$2b_1 + b_2 = -\frac{1}{4}$. Choose $b_1 = 0$, $b_2 = -\frac{1}{4}$

$$x_p = t\left(-\tfrac{1}{4}\right)\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} + \begin{pmatrix}0\\-\frac{1}{4}\end{pmatrix}e^{-t}$$

$$= t\begin{pmatrix}-\frac{1}{4}\\ \frac{1}{2}\end{pmatrix}e^{-t} + \begin{pmatrix}0\\-\frac{1}{4}\end{pmatrix}e^{-t}$$

#

Ex. 2. Use the Method of Variation of Parameters to solve

$$x' = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}x + \begin{pmatrix}0\\1\end{pmatrix}e^{-t}$$

Sol'n: $x^{(1)} = \begin{pmatrix}1\\2\end{pmatrix}e^{3t} = \begin{pmatrix}e^{3t}\\2e^{3t}\end{pmatrix}$, $x^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix}e^{-t} = \begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix}$

Fundamental Matrix is

$$\Psi(t) = \left(x^{(1)} \ \ x^{(2)}\right) = \begin{pmatrix}e^{3t} & e^{-t}\\2e^{3t} & -2e^{-t}\end{pmatrix}$$

$x_p = \Psi(t)\vec{c}(t)$, where $\vec{c} = \begin{pmatrix}c_1(t)\\c_2(t)\end{pmatrix}$. Then

$$\Psi(t)\vec{c}\,'(t) = g(t)$$

$$\begin{pmatrix}e^{3t} & e^{-t}\\2e^{3t} & -2e^{-t}\end{pmatrix}\begin{pmatrix}c_1'\\c_2'\end{pmatrix} = \begin{pmatrix}0\\e^{-t}\end{pmatrix} \Rightarrow \begin{cases} e^{3t}c_1' + e^{-t}c_2' = 0 & \text{--(1)}\\ 2e^{3t}c_1' - 2e^{-t}c_2' = e^{-t} & \text{--(2)}\end{cases}$$

From (1) $\Rightarrow$

$$c_2' = -e^{-4t} c_1'$$

Substituting into (2) $\Rightarrow$

$$2e^{3t}c_1' - 2e^{-t}(-e^{-4t}c_1') = e^{-t}$$

$$4e^{3t}c_1' = e^{-t}$$

$$c_1' = \frac{1}{4}e^{-4t}$$

$$c_1 = \frac{-1}{16}e^{-4t}$$

So $c_2' = -e^{-4t}c_1' = -\frac{1}{4} \Rightarrow c_2 = -\frac{1}{4}t$

So $$X_p = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}\begin{pmatrix} -\frac{1}{16}e^{-4t} \\ -\frac{1}{4}t \end{pmatrix}$$

$$= -\frac{1}{16}e^{-4t}\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} - \frac{1}{4}t\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$$

$$= t\begin{pmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{pmatrix}e^{-t} + \begin{pmatrix} -\frac{1}{16} \\ -\frac{1}{8} \end{pmatrix}e^{-t}$$

Ex. 3 Use the Method of Diagonalization to solve

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}$$

Sol'n: $x^{(1)} = \begin{pmatrix}1\\2\end{pmatrix}e^{3t}$, $x^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix}e^{-t}$

so $a^{(1)} = \begin{pmatrix}1\\2\end{pmatrix}$, $a^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix}$, $r_1 = 3$, $r_2 = -1$

$$T = \begin{pmatrix}a^{(1)} & a^{(2)}\end{pmatrix} = \begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}$$

$x_p = Ty \Rightarrow$

$$y' = \begin{pmatrix}r_1 & 0\\ 0 & r_2\end{pmatrix}y + T^{-1}g, \quad r_1 = 3,\ r_2 = -1$$

$$\boxed{\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}}$$

$$T^{-1} = \frac{1}{-4}\begin{pmatrix}-2 & -1\\ -2 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & -\frac{1}{4}\end{pmatrix}$$

$$T^{-1}g = \begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & -\frac{1}{4}\end{pmatrix}\begin{pmatrix}0\\ e^{-t}\end{pmatrix} = \begin{pmatrix}\frac{1}{4}\\ -\frac{1}{4}\end{pmatrix}e^{-t}$$

$$y_1' = r_1 y_1 + h_1 \Rightarrow y_1' = 3y_1 + \frac{1}{4}e^{-t} \quad -(1)$$

$$y_2' = r_2 y_2 + h_2 \Rightarrow y_2' = -y_2 - \frac{1}{4}e^{-t} \quad -(2)$$

To solve (1), we use method of undetermined coefficients.

$y_1 = Ae^{-t} \Rightarrow -A = 3A + \frac{1}{4} \Rightarrow A = -\frac{1}{16}$

$y_2 = Bte^{-t} \Rightarrow B = -\frac{1}{4}$

so $y = \begin{pmatrix}-\frac{1}{16}e^{-t}\\ -\frac{1}{4}te^{-t}\end{pmatrix}$, $x_p = Ty = \begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}-\frac{1}{16}e^{-t}\\ -\frac{1}{4}te^{-t}\end{pmatrix}$

$$x_p = -\frac{1}{4}te^{-t}\begin{pmatrix}1\\-2\end{pmatrix} - \frac{1}{16}e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

$$= te^{-t}\begin{pmatrix}-\frac{1}{4}\\ \frac{1}{2}\end{pmatrix} + e^{-t}\begin{pmatrix}-\frac{1}{16}\\ -\frac{1}{8}\end{pmatrix} \quad \#$$

Remark:
- Ex. 2 & 3 give the same sol'n
- x_p in Ex. 1 $-$ x_p in Ex. 2 $= \begin{pmatrix}\frac{1}{16}\\ -\frac{1}{8}\end{pmatrix}e^{-t} = \frac{1}{16}\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} = \frac{1}{16}x^{(2)}$

\#