

Be sure this exam has 10 pages including the cover

The University of British Columbia

MATH 256, Section 103

Midterm Exam II — November 16, 2018

Name \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

This exam consists of 5 questions. No notes. Simple numerics calculators are allowed. List of Laplace Transform is provided. Write your answer in the blank page provided.

Problem	max score	score
1.	15	
2.	25	
3.	20	
4.	20	
5.	20	
total	100	

**1. Each candidate should be prepared to produce his library/AMS card upon request.**

**2. Read and observe the following rules:**

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

**CAUTION** - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

**3. Smoking is not permitted during examinations.**

(15 points) 1. This question contains three parts. For each question, **multiple choices are allowed**.

(5 points) (a) The characteristic root equation for a tenth order linear homogeneous ordinary differential equation with constant coefficients is given by

$$(r - 2)^3(r + 1)((r - 1)^2 + 1)^3 = 0.$$

Which of the followings are solutions to the differential equation?

(A)  $t^3e^{2t}$ ; (B)  $te^{2t}$ ; (C)  $t^2e^t \sin t$ ; (D)  $e^t$ ; (E)  $t^3e^t \cos t$ ; (F)  $te^t$ ; (G)  $e^{-t}$

Your answer is ( **B, C, G** )

(5 points) (b) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{3t} \\ t \end{pmatrix}.$$

For the method of undetermined coefficients which of the followings is the form of the special solution?

(A)  $\vec{a}e^{3t} + \vec{b}t$ ; (B)  $t\vec{a}e^{3t} + \vec{b}t + \vec{c}$ ; (C)  $\vec{a}e^{3t} + t\vec{b} + \vec{c}$ ; (D)  $t\vec{a}e^{3t} + \vec{b}e^{3t} + t\vec{c} + \vec{d}$ ; (E) None of the above

Your answer is ( **D** )

(5 points) (c) Consider the partial fractions of  $\frac{8s^2+6s+1}{(s^2+1)^2(s+1)^2s^3(s-1)}$ . Which of the followings are part of the partial fractions decomposition?

(A)  $\frac{As+B}{s^2+1}$ ; (B)  $\frac{Cs}{(s+1)^2}$ ; (C)  $\frac{D}{s^2}$ ; (D)  $\frac{E}{s-1}$ ; (E)  $\frac{Fs^2+G}{(s^2+1)^2}$ ; (F)  $\frac{Hs+I}{(s^2+1)^2}$ ; (G)  $\frac{J}{s+1}$

Your answer is ( **A, C, D, F, G** )

- (25 points) 2. For the following system of ordinary differential equation, (a) find the general solutions; (b) classify the types (saddle, node (source or sink), spirals) and the stability; (c) draw a few trajectories

$$(1) \mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix} \mathbf{x},$$

$$(2) \mathbf{x}' = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix} \mathbf{x}$$

10

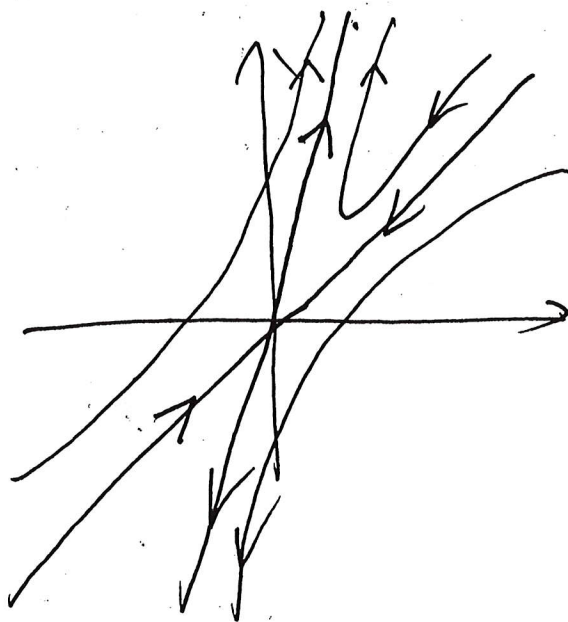
$$(1) \begin{vmatrix} -2-r & 1 \\ -3 & 2-r \end{vmatrix} = 0 \Rightarrow r^2 - 4 + 3 = 0 \Rightarrow r = \pm 1$$

$$r_1 = -1 \Rightarrow \begin{pmatrix} -2+1 & 1 \\ -3 & 2+1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r_2 = 1 \Rightarrow \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\mathbf{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

saddle, unstable



$$(2) \begin{vmatrix} -3-r & -1 \\ 5 & 1-r \end{vmatrix} = (r+3)(r-1) + 5$$

$$= r^2 + 2r + 2 = 0$$

$$r = -1 \pm i, \quad r_1 = -1 + i$$

$$\begin{pmatrix} -2-i & -1 \\ 5 & 2-i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = \begin{pmatrix} 1 \\ -2-i \end{pmatrix}$$

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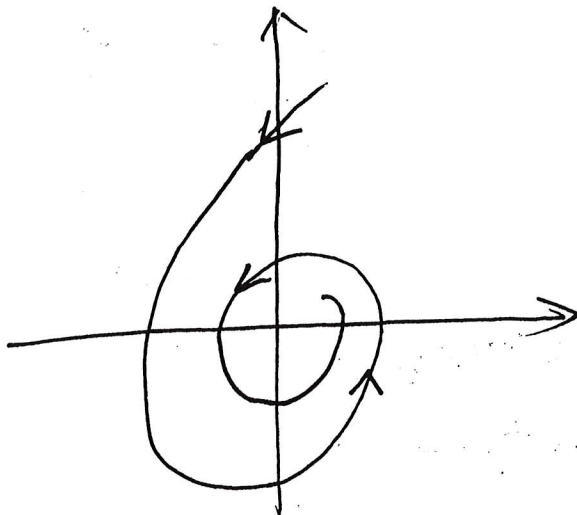
$$e^{-t+it} \begin{pmatrix} 1 \\ -2-i \end{pmatrix} = \begin{pmatrix} e^{-t} \cos t + i e^{-t} \sin t \\ -(2+i)(e^{-t} \cos t + i e^{-t} \sin t) \end{pmatrix}$$

$$\begin{pmatrix} e^{-t} \cos t \\ -2e^{-t} \cos t + e^{-t} \sin t \end{pmatrix} + i \begin{pmatrix} e^{-t} \sin t \\ -2e^{-t} \sin t - e^{-t} \cos t \end{pmatrix}$$

$$X = c_1 \begin{pmatrix} e^{-t} \cos t \\ -2e^{-t} \cos t + e^{-t} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \sin t \\ -2e^{-t} \sin t - e^{-t} \cos t \end{pmatrix}$$

$\lambda = -1 \Rightarrow$  spiral, ~~stable~~

To see the trajectory,  $x^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow x' = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow x_1' = -1$   
 $\Rightarrow x_1$  decreasing



counter-clockwise

(20 points) 3. Use the method of variation of parameters to obtain the general solutions of

$$x' = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} x + e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hint: The eigenvalues are repeated  $r_1 = r_2 = -1$  with eigenvector given by  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

Sol'n:  $x^{(1)} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$x^{(2)} = t e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-t} b$$

Then  $Pb = -b + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$2b_1 + 2b_2 = -1, \quad b = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} x^{(1)} & x^{(2)} \end{pmatrix}$$

By the method of undetermined coefficients:  $x = x_p = \Psi(t)c(t)$

$$\Psi c'(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} e^{-t} & t e^{-t} - \frac{1}{2} e^{-t} \\ -e^{-t} & -t e^{-t} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

$$e^{-t} c_1' + (t e^{-t} - \frac{1}{2} e^{-t}) c_2' = e^{-t} \Rightarrow c_1' + t c_2' - \frac{1}{2} c_2' = 1$$

$$-e^{-t} c_1' - t e^{-t} c_2' = 0 \Rightarrow c_1' + t c_2' = 0$$

$$c_2' = -2 \Rightarrow c_2 = -2t, \quad c_1' = 2t \Rightarrow c_1 = t^2$$

$$x_p = \begin{pmatrix} e^{-t} & t e^{-t} - \frac{1}{2} e^{-t} \\ -e^{-t} & -t e^{-t} \end{pmatrix} \begin{pmatrix} t^2 \\ -2t \end{pmatrix}$$

$$x = x_p + c_1 x^{(1)} + c_2 x^{(2)}$$

(20 points) 4. This problem contains two parts.

(10 points) (a) Write the following function

$$g(t) = \begin{cases} 1, & 0 \leq t < \pi \\ \sin(t), & \pi \leq t < +\infty \end{cases}$$

as linear sums of  $H(t-c)$ ,  $H(t-c)f(t-c)$ . Then use the list of Laplace transform formulas to find its Laplace transform.

(10 points) (b) Compute the Laplace transform of the following functions

(A)  $H(t-1)e^t$ ; (B)  $\delta(t-1)e^t$ ; (C)  $\int_0^t e^{\tau-t} \sin(\tau) d\tau$

(List of Laplace transform formulas is attached at the last page)

$$\begin{aligned} \text{(a)} \quad g(t) &= 1 + (\sin t - 1) H(t-\pi) \\ &= 1 + (\sin(t-\pi+\pi) - 1) H(t-\pi) \\ &= 1 - (1 + \sin(t-\pi)) H(t-\pi) \end{aligned}$$

$$L[g](s) = \frac{1}{s} - \left( \frac{1}{s} + \frac{1}{s^2+1} \right) e^{-\pi s}$$

$$\begin{aligned} \text{(b)} \quad L[H(t-1)e^t] &= L[H(t-1)e^{t-1}] e = e \cdot \frac{e^{-s}}{s-1} \\ L[\delta(t-1)e^t] &= L[\delta(t-1)e] = e \cdot e^{-s} \\ L\left[\int_0^t e^{\tau-t} \sin \tau d\tau\right] &= L\left[\int_0^t e^{-(t-\tau)} \sin \tau d\tau\right] \\ &= L[e^{-t}] L[\sin t] \\ &= \frac{1}{s+1} \cdot \frac{1}{s^2+1} \end{aligned}$$

(20 points) 5. The Laplace transform of the solution to a differential equation is given by

$$Y(s) = \frac{s-2}{s^2+4} + \frac{2e^{-s}}{(s^2+1)(s^2+4)} + \frac{e^{-2s}}{s^2+4}$$

(a) Find the solution  $y(t)$  by inverting  $Y(s)$ . Hint:

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3} \frac{1}{s^2+1} - \frac{1}{3} \frac{1}{s^2+4}$$

(b) Provide a second-order differential equation of the form

$$y'' + by' + cy = g(t), y(0) = y_0, y'(0) = y_1$$

for  $y(t)$  and initial values for  $y(0)$  and  $y'(0)$  that has a transformed solution given by  $Y(s)$ . You may use  $H(t-c)$  and  $\delta(t-c)$ .

(List of Laplace transform formulas is attached at the last page)

(a)  $Y(s) = \frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{2}{3} e^{-s} \frac{1}{s^2+1} - \frac{2}{3} e^{-s} \frac{1}{s^2+4} + \frac{e^{-2s}}{s^2+4}$

$y(t) = \cos 2t - \sin 2t + \frac{2}{3} H(t-1) \sin(t-1) - \frac{1}{3} H(t-1) \sin 2(t-1) + \frac{1}{2} H(t-2) \sin 2(t-1)$

(b) The first term is  $\frac{s-2}{s^2+4}$ . So

$$y'' + 4y = g(t)$$

$$\Rightarrow s^2 Y(s) + 4Y(s) - s y(0) - y'(0) = L[g]$$

$$Y = \frac{s y(0) + y'(0)}{s^2 + 4} + \frac{L[g]}{s^2 + 4}$$

$$y(0)=1, y'(0)=-2, L[g] = \frac{2e^{-s}}{s^2+1} + e^{-2s}$$

$$g = 2H(t-1) \sin(t-1) + \delta(t-2)$$

$$\begin{cases} y'' + 4y = 2H(t-1) \sin(t-1) + \delta(t-2) \\ y(0)=1, y'(0)=-2 \end{cases}$$