Be sure this exam has 10 pages including the cover

The University of British Columbia

MATH 256, Section 103

Midterm Exam II —November 16, 2018

Name ______________________ Signature ______________________

Student Number ______________________

This exam consists of 5 questions. No notes. Simple numerics calculators are allowed. List of Laplace Transform is provided. Write your answer in the blank page provided.

Problem	max score	score
1.	15	
2.	25	
3.	20	
4.	20	
5.	20	
total	100	

(15 points) 1. This question contains three parts. For each question, **multiple choices are allowed.**

(5 points) (a) The characteristic root equation for a tenth order linear homogeneous ordinary differential equation with constant coefficients is given by

$$(r-2)^3(r+1)\left((r-1)^2+1\right)^3 = 0.$$

Which of the followings are solutions to the differential equation?
(A) t^3e^{2t}; (B) te^{2t}; (C) $t^2e^t\sin t$; (D) e^t; (E) $t^3e^t\cos t$; (F) te^t; (G) e^{-t}

Your answer is (B, C, G)

(5 points) (b) Consider the following system

$$x' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} e^{3t} \\ t \end{pmatrix}.$$

For the method of undetermined coefficients which of the followings is the form of the special solution?
(A) $\vec{a}e^{3t} + \vec{b}t$; (B) $t\vec{a}e^{3t} + \vec{b}t + \vec{c}$; (C) $\vec{a}e^{3t} + t\vec{b} + \vec{c}$; (D) $t\vec{a}e^{3t} + \vec{b}e^{3t} + t\vec{c} + \vec{d}$; (E) None of the above

Your answer is (D)

(5 points) (c) Consider the partial fractions of $\frac{8s^2+6s+1}{(s^2+1)^2(s+1)^2s^3(s-1)}$. Which of the followings are part of the partial fractions decomposition?

(A) $\frac{As+B}{s^2+1}$; (B) $\frac{Cs}{(s+1)^2}$; (C) $\frac{D}{s^2}$; (D) $\frac{E}{s-1}$; (E) $\frac{Fs^2+G}{(s^2+1)^2}$; (F) $\frac{Hs+I}{(s^2+1)^2}$; (G) $\frac{J}{s+1}$

Your answer is (A, C, D, F, G)

(25 points) 2. For the following system of ordinary differential equation, (a) find the general solutions; (b) classify the types (saddle, node (source or sink), spirals) and the stability; (c) draw a few trajectories

$$(1)\ \mathrm{x}' = \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix} \mathrm{x}, \qquad (2)\ \mathrm{x}' = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix} \mathrm{x}$$

10

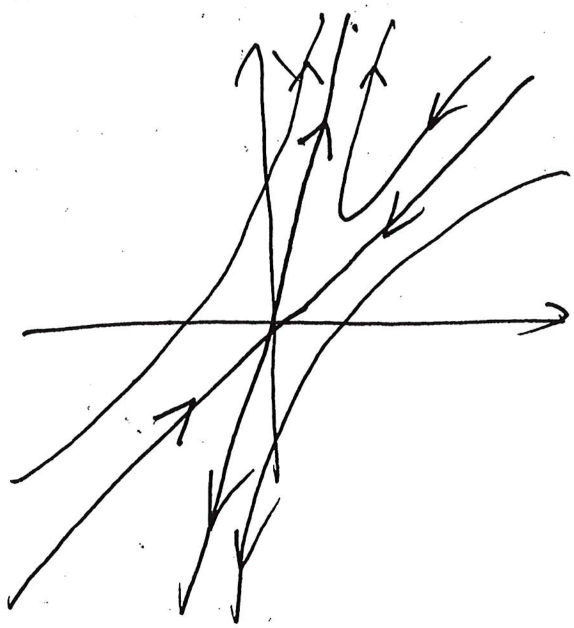
(1) $\begin{vmatrix} -2-r & 1 \\ -3 & 2-r \end{vmatrix} = 0 \Rightarrow r^2-4+3=0 \Rightarrow r = \pm 1$

$r_1 = -1 \Rightarrow \begin{pmatrix} -2+1 & 1 \\ -3 & 2+1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_2 = 1 \Rightarrow \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$X = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

saddle, unstable



$\Leftarrow$ 2

(2). $\begin{vmatrix} -3-r & -1 \\ 5 & 1-r \end{vmatrix} = (r+3)(r-1)+5$

$= r^2+2r+2 = 0$

$r = -1 \pm i, \quad r_1 = -1+i$

$\begin{pmatrix} -2-i & -1 \\ 5 & 2-i \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = \begin{pmatrix} 1 \\ -2-i \end{pmatrix}$

(continuing blank page)

$$e^{-t+it}\begin{pmatrix}1\\-2-i\end{pmatrix} = \begin{pmatrix}e^{-t}\cos t + ie^{-t}\sin t\\ -(2+i)(e^{-t}\cos t + ie^{-t}\sin t)\end{pmatrix}$$

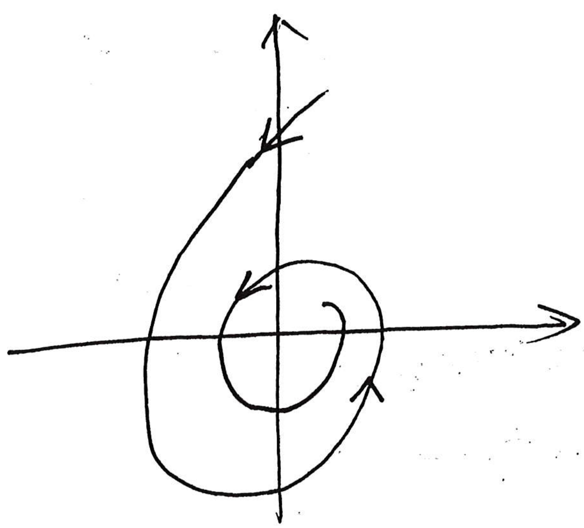
$$= \begin{pmatrix}e^{-t}\cos t\\ -2e^{-t}\cos t + e^{-t}\sin t\end{pmatrix} + i\begin{pmatrix}e^{-t}\sin t\\ -2e^{-t}\sin t - e^{-t}\cos t\end{pmatrix}$$

$$X = c_1\begin{pmatrix}e^{-t}\cos t\\ -2e^{-t}\cos t + e^{-t}\sin t\end{pmatrix} + c_2\begin{pmatrix}e^{-t}\sin t\\ -2e^{-t}\sin t - e^{-t}\cos t\end{pmatrix}$$

$\lambda = -1 \Rightarrow$ spiral, stable

To see the trajectory, $x^0 = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow X' = \begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow x_1' = -1$

$\Rightarrow x_1$ decreasing



counter-clockwise

(20 points) 3. Use the method of variation of parameters to obtain the general solutions of

$$\mathbf{x}' = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hint: The eigenvalues are repeated $r_1 = r_2 = -1$ with eigenvector given by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Sol'n: $x^{(1)} = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x^{(2)} = t e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-t} b$

Then $Pb = -b + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$2b_1 + 2b_2 = -1$, $b = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$

+ 8

$\Psi(t) = \begin{pmatrix} x^{(1)} & x^{(2)} \end{pmatrix}$

By the method of undetermined coefficients: $x = x_p = \Psi(t) c(t)$

$\Psi c'(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (+3) $\Rightarrow$

$$\begin{pmatrix} e^{-t} & te^{-t} - \frac{1}{2}e^{-t} \\ -e^{-t} & -te^{-t} \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

$e^{-t}c_1' + (te^{-t} - \frac{1}{2}e^{-t})c_2' = e^{-t} \Rightarrow c_1' + tc_2' - \frac{1}{2}c_2' = 1$

$-e^{-t}c_1' - te^{-t}c_2' = 0 \Rightarrow c_1' + tc_2' = 0$

$c_2' = -2 \Rightarrow c_2 = -2t$, $c_1' = 2t \Rightarrow c_1 = t^2$ (+9)

$$x_p = \begin{pmatrix} e^{-t} & te^{-t} - \frac{1}{2}e^{-t} \\ -e^{-t} & -te^{-t} \end{pmatrix}\begin{pmatrix} t^2 \\ -2t \end{pmatrix}$$

$x = x_p + c_1 x^{(1)} + c_2 x^{(2)}$

(20 points) 4. This problem contains two parts.

(10 points) (a) Write the following function

$$g(t) = \begin{cases} 1, & 0 \le t < \pi \\ \sin(t), & \pi \le t < +\infty \end{cases}$$

as linear sums of $H(t-c), H(t-c)f(t-c)$. Then use the list of Laplace transform formulas to find its Laplace transform.

(10 points) (b) Compute the Laplace transform of the following functions
(A) $H(t-1)e^t$; (B) $\delta(t-1)e^t$; (C) $\int_0^t e^{\tau - t}\sin(\tau)d\tau$

(List of Laplace transform formulas is attached at the last page)

(a) $g(t) = 1 + (\sin t - 1)\, H(t-\pi)$

$= 1 + (\sin(t-\pi+\pi) - 1)\, H(t-\pi)$

$= 1 - (1 + \sin(t-\pi))\, H(t-\pi)$

$L[g](s) = \frac{1}{s} - \left(\frac{1}{s} + \frac{1}{s^2+1}\right) e^{-\pi s}$

(b) $L[H(t-1)e^t] = L[H(t-1)e^{t-1}]\, e = e \cdot \frac{e^{-s}}{s-1}$

$L[\delta(t-1)e^t] = L[\delta(t-1)\, e] = e \cdot e^{-s}$

$L\left[\int_0^t e^{\tau-t} \sin\tau\, d\tau\right] = L\left[\int_0^t e^{-(t-\tau)} \sin\tau\, d\tau\right]$

$= L[e^{-t}]\, L[\sin t]$

$= \frac{1}{s+1} \cdot \frac{1}{s^2+1}$

(20 points) 5. The Laplace transform of the solution to a differential equation is given by

$$Y(s) = \frac{s-2}{s^2+4} + \frac{2e^{-s}}{(s^2+1)(s^2+4)} + \frac{e^{-2s}}{s^2+4}.$$

(a) Find the solution $y(t)$ by inverting $Y(s)$. Hint:

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}.$$

(b) Provide a second-order differential equation of the form

$$y'' + by' + cy = g(t), y(0) = y_0, y'(0) = y_1$$

for $y(t)$ and initial values for $y(0)$ and $y'(0)$ that has a transformed solution given by $Y(s)$. You may use $H(t-c)$ and $\delta(t-c)$.

(List of Laplace transform formulas is attached at the last page)

+10

(a) $Y(s) = \frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{2}{3}e^{-s}\frac{1}{s^2+1} - \frac{2}{3}e^{-s}\frac{1}{s^2+4} + \frac{e^{-2s}}{s^2+4}$

$y(t) = \cos 2t - \sin 2t + \frac{2}{3}H(t-1)\sin(t-1) - \frac{1}{3}H(t-1)\sin 2(t-1)$

$+ \frac{1}{2}H(t-2)\sin 2(t-1)$

+10

(b) The first term is $\frac{s-2}{s^2+4}$. So

$y'' + 4y = g(t)$

$\Rightarrow s^2 Y(s) + 4Y(s) - s\,y(0) - y'(0) = L[g]$

$Y = \frac{s y(0) + y'(0)}{s^2+4} + \frac{L[g]}{s^2+4}$

$y(0) = 1, y'(0) = -2, \quad L[g] = \frac{2e^{-s}}{s^2+1} + e^{-2s}$

$g = 2H(t-1)\sin(t-1) + \delta(t-2)$

$$\begin{cases} y'' + 4y = 2H(t-1)\sin(t-1) + \delta(t-2) \\ y(0) = 1, y'(0) = -2 \end{cases}$$