

Be sure this exam has 8 pages including the cover

The University of British Columbia

MATH 256, Section 103

Midterm Exam I – October 12 2018

Name _____ Signature _____

Student Number _____

This exam consists of 5 questions. No notes. Simple numerics calculators are allowed. Write your answer in the blank page provided.

Problem	max score	score
1.	18	
2.	12	
3.	20	
4.	25	
5.	25	
total	100	

1. Each candidate should be prepared to produce his library/AMS card upon request.
2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

 - (a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
 - (b) Speaking or communicating with other candidates.
 - (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
3. Smoking is not permitted during examinations.

(18 points) 1. Solve the following ordinary differential equation

$$ty' - (t+1)y = e^{-t}y^2, \quad y(1) = 1$$

and state the Interval of Existence.

Hint: let $v = y^{-1} = \frac{1}{y}$.

Solutions: ~~let~~ $y' - \frac{t+1}{t}y = \frac{1}{t}e^{-t}y^2$

$$v = y^{-1} \Rightarrow$$

$$v' + \frac{t+1}{t}v = -\frac{1}{t}e^{-t}$$

$$\mu = e^{\int \frac{t+1}{t} dt} = te^t$$

$$\int \mu g = \int te^t \left(-\frac{1}{t}e^{-t}\right) dt = -t$$

$$v = \frac{1}{te^t} (c-t)$$

$$v(1) = 1 \Rightarrow e = c-1 \Rightarrow c = 1+e$$

$$v = \frac{1}{te^t} (1+e-t)$$

$$y = \frac{te^t}{1+e-t}$$

Interval of Existence: $t \neq 0, t \neq 1+e \Rightarrow$

$$0 < t < 1+e$$

(12 points) 2. This problem contains two parts.

(4 points) (a) Which of the following ordinary differential equations may have nonunique solutions?

- (A) $y' = y^{1/3}, y(0) = 1$; (B) $y' = y^{1/3}, y(0) = 0$;
 (C) $y' = y^{5/3}, y(0) = 1$; (D) $y' = y^{5/3}, y(0) = 0$

Your answer is (B)

(8 points) (b) Consider the following ordinary differential equation

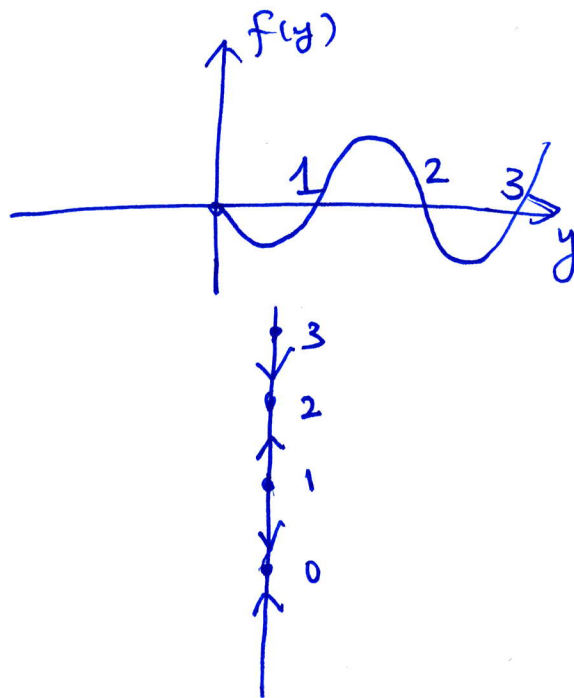
$$\frac{dy}{dt} = y(y-1)(y-2)(y-3), \quad y(0) = y_0$$

(b1) If $y_0 = 0.5$, what is $\lim_{t \rightarrow +\infty} y(t)$?

(b2) If $y_0 = 1.5$, what is $\lim_{t \rightarrow +\infty} y(t)$?

(b3) If $y_0 = 2.5$, what is $\lim_{t \rightarrow +\infty} y(t)$?

(b4) If $y_0 = 3$, what is $\lim_{t \rightarrow +\infty} y(t)$?



(b1) $\lim_{t \rightarrow +\infty} y(t) = 0$

(b2) $\lim_{t \rightarrow +\infty} y(t) = 2$

(b3) $\lim_{t \rightarrow +\infty} y(t) = 2$

(b4) $y(t) = y_0 = 3$ so $\lim_{t \rightarrow +\infty} y(t) = 3$

(20 points) 3. Consider the following second order ordinary differential equation:

$$y'' - 2y' + 2y = h(t)$$

(5 points) (a) Find the solutions to the homogeneous problem

$$y'' - 2y' + 2y = 0.$$

(10 points) (b) Suppose $h(t) = 5 \cos(t)$. Use the method of undetermined coefficients to find a special solution to the inhomogeneous problem.

(5 points) (c) Suppose $h(t) = te^t \sin(t) - t^2 + 2e^{-t}$. Use the method of undetermined coefficients to find the form of the special solution y_p . Do not attempt to find the coefficients.

Solutions:

(a). $r^2 - 2r + 2 = 0$, $r = 1 \pm i$
 $y_1 = e^t \cos t$, $y_2 = e^t \sin t$

(b) $y_p = t^s [A \cos t + B \sin t]$, $s = 0$.

$$y_p' = -A \sin t + B \cos t$$

$$y_p'' = -A \cos t - B \sin t = -y_p$$

$$\begin{aligned} y_p'' - 2y_p' + 2y_p &= -y_p - 2(-A \sin t + B \cos t) + 2y_p \\ &= A \cos t + B \sin t + 2A \sin t - 2B \cos t = 5 \cos t \end{aligned}$$

$$A - 2B = 5$$

$$2A + B = 0$$

$$\Rightarrow A = 1, B = -2$$

$$y_p = \cos t - 2 \sin t$$

(c). $y_p = t^{s_1} [(At+B) \cos t e^t + (Ct+D) \sin t e^t]$
 $+ t^{s_2} [Et^2 + Ft + G]$
 $+ t^{s_3} H e^{-t}$

$$s_1 = 1, s_2 = 0, s_3 = 0$$

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(25 points) 4. Consider the following second order ODE:

$$2t^2 y'' - ty' + y = 3t^2, t > 0$$

(5 points) (a) Compute the Wronskian $W(t)$.

(10 points) (b) Find the solutions to the homogeneous problem:

$$2t^2 y'' - ty' + y = 0, t > 0.$$

(10 points) (c) Use the method of variation of parameters to find the special solution y_p to the inhomogeneous problem

$$2t^2 y'' - ty' + y = 3t^2, t > 0$$

Solutions

(a) $y'' - \frac{1}{2t} y' + \frac{1}{2t^2} y = \frac{3}{2}$
 $p = -\frac{1}{2t}$, $W = c e^{-\int p t dt} = c e^{\int \frac{1}{2t} dt} = c \sqrt{t}$

(b) Euler type, $y = t^r$
 $2r(r-1) - r + 1 = 0 \Rightarrow 2r^2 - 3r + 1 = 0 \Rightarrow r_1 = \frac{1}{2}, r_2 = 1$

$$y = c_1 t^{\frac{1}{2}} + c_2 t$$

(c) Let $y_p = u_1 t + u_2 t^{\frac{1}{2}}$

$$u_1' t + u_2' t^{\frac{1}{2}} = 0 \Rightarrow u_2' = -u_1' t^{\frac{1}{2}}$$

$$u_1' + \frac{1}{2} u_2' t^{-\frac{1}{2}} = \frac{3}{2}$$

$$u_1' - \frac{1}{2} u_1' = \frac{3}{2} \Rightarrow u_1' = 3 \Rightarrow u_1 = 3t$$

$$u_2' = -3t^{\frac{1}{2}}$$

$$u_2 = -2t^{\frac{3}{2}}$$

$$y_p = 3t \cdot t - 2t^{\frac{3}{2}} t^{\frac{1}{2}} = t^2$$

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(25 points) 5. A mass weighing 16 lb stretches a spring 2 ft by itself.

(15 points) (a) Suppose that there is no damping and no external forces acting on the system. The spring is initially displaced 3 in upwards from its equilibrium position and given an initial velocity of 1 ft/sec downward. Find the displacement $u(t)$ at any time t . Write the solution in the form $R \cos(\omega_0 t - \delta)$. ($R > 0, \omega_0 > 0, 0 \leq \delta < 2\pi$.)

Hint: 1 ft = 12 in. $g = 32 \text{ ft/sec}^2$. $m = \frac{w}{g}, k = \frac{w}{l}$.

(10 points) (b) Suppose that the external force $F_e = \cos(\omega t)$ is added. Let the damping coefficient be γ . (Here $\gamma = 0$ or $\gamma > 0$.) Under what conditions on γ and ω does the displacement remain bounded at $t \rightarrow +\infty$? State your reason.

5 (a). $m = \frac{16}{32} = \frac{1}{2}$

$k = \frac{16}{2} = 8$

$\gamma = 0$

$u(0) = 3 \text{ in} = -\frac{1}{4}$

$u'(0) = 1$

$m u'' + \gamma u' + k u = 0 \Rightarrow \begin{cases} \frac{1}{2} u'' + 8 u = 0 \\ u(0) = -\frac{1}{4}, u'(0) = 1 \end{cases}$

$\omega_0 = \sqrt{\frac{k}{m}} = 4$

$u = A \cos 4t + B \sin 4t$

$u = -\frac{1}{4} \cos 4t + \frac{1}{4} \sin 4t$

$A = -\frac{1}{4}, B = \frac{1}{4} \Rightarrow R = \sqrt{\left(-\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{\sqrt{2}}{4}$

$\frac{\sqrt{2}}{4} \cos \delta = -\frac{1}{4} \Rightarrow \cos \delta = -\frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}}{4} \sin \delta = \frac{1}{4} \Rightarrow \sin \delta = \frac{\sqrt{2}}{2}$

$u = \frac{\sqrt{2}}{4} \cos\left(4t - \frac{3\pi}{4}\right)$

(b). If either $\gamma > 0$ or $\gamma = 0, \omega \neq \omega_0$, then $u(t)$ remains bounded.

Reason: $\gamma > 0, y = y_h + y_p = y_h + A \cos \omega t, y_h \rightarrow 0 \text{ as } t \rightarrow \infty$
 $\gamma = 0, y = A \cos \omega t + B \sin \omega t + A \cos \omega t, \text{ if } \omega \neq \omega_0.$