Be sure this exam has 8 pages including the cover

The University of British Columbia

MATH 256, Section 103

Midterm Exam I – October 12 2018

Name ______________________ Signature ______________________

Student Number ______________________

This exam consists of **5** questions. No notes. Simple numerics calculators are allowed. Write your answer in the blank page provided.

| Problem | max score | score |
|---|---|---|
| 1. | 18 | |
| 2. | 12 | |
| 3. | 20 | |
| 4. | 25 | |
| 5. | 25 | |
| total | 100 | |

(18 points) 1. Solve the following ordinary differential equation

$$ty' - (t+1)y = e^{-t}y^2, \qquad y(1) = 1$$

and state the Interval of Existence.

Hint: let $v = y^{-1} = \frac{1}{y}$.

Solutions: ~~Let v~~ $y' - \frac{t+1}{t} y = \frac{1}{t} e^{-t} y^2$

$v = y^{-1} \Rightarrow$

$$v' + \frac{t+1}{t} v = -\frac{1}{t} e^{-t},$$

$$\mu = e^{\int \frac{t+1}{t} dt} = te^t$$

$$\int \mu g = \int te^t \left(-\frac{1}{t} e^{-t}\right) dt = -t$$

$$v = \frac{1}{te^t}(c-t)$$

$v(1) = 1 \Rightarrow \quad e = c - 1 \Rightarrow c = 1+e$

$$v = \frac{1}{te^t}(1+e-t)$$

$$y = \frac{te^t}{1+e-t}$$

Interval of Existence: $t \neq 0, \ t \neq 1+e \Rightarrow$

$$0 < t < 1+e$$

(12 points) 2. This problem contains two parts.

(4 points) (a) Which of the following ordinary differential equations may have nonunique solutions?

(A) $y' = y^{1/3}, y(0) = 1$; (B) $y' = y^{1/3}, y(0) = 0$;

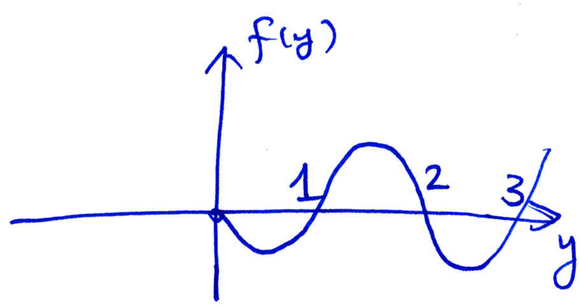
(C) $y' = y^{5/3}, y(0) = 1$; (D) $y' = y^{5/3}, y(0) = 0$

Your answer is ( B )

(8 points) (b) Consider the following ordinary differential equation

$$\frac{dy}{dt} = y(y-1)(y-2)(y-3), \qquad y(0) = y_0$$

(b1) If $y_0 = 0.5$, what is $\lim_{t\to+\infty} y(t)$?




(b2) If $y_0 = 1.5$, what is $\lim_{t\to+\infty} y(t)$?

(b3) If $y_0 = 2.5$, what is $\lim_{t\to+\infty} y(t)$?

(b4) If $y_0 = 3$, what is $\lim_{t\to+\infty} y(t)$?


(b1) $\lim_{t\to+\infty} y(t) = 0$

(b2) $\lim_{t\to+\infty} y(t) = 2$

(b3) $\lim_{t\to+\infty} y(t) = 2$

(b4) $y(t) \equiv y_0 = 3$ So $\lim_{t\to+\infty} y(t) = 3$

(20 points) 3. Consider the following second order ordinary differential equation:

$$y'' - 2y' + 2y = h(t)$$

(5 points) (a) Find the solutions to the homogeneous problem

$$y'' - 2y' + 2y = 0.$$

(10 points) (b) Suppose $h(t) = 5\cos(t)$. Use the method of undetermined coefficients to find a special solution to the inhomogeneous problem.

(5 points) (c) Suppose $h(t) = te^t \sin(t) - t^2 + 2e^{-t}$. Use the method of undetermined coefficients to find the form of the special solution $y_p$. Do not attempt to find the coefficients.

Solutions:

(a). $r^2 - 2r + 2 = 0, \quad r = 1 \pm i$

$y_1 = e^t \cos t, \quad y_2 = e^t \sin t$

(b) $y_p = t^s [A\cos t + B\sin t] \quad s = 0.$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t = -y_p$

$y_p'' - 2y_p' + 2y_p = -y_p - 2(-A\sin t + B\cos t) + 2y_p$

$= A\cos t + B\sin t + 2A\sin t - 2B\cos t = 5\cos t$

$A - 2B = 5$
$2A + B = 0$
$\Rightarrow A = 1, \ B = -2$

$y_p = \cos t - 2\sin t$

(c). $y_p = t^{s_1} [(At+B)\cos t \, e^t + (Ct+D)\sin t \, e^t]$

$+ t^{s_2} [Et^2 + Ft + G]$

$+ t^{s_3} He^{-t}$

$s_1 = 1, \quad s_2 = 0, \quad s_3 = 0$

(continuing blank page)

(25 points) 4. Consider the following second order ODE:

$$2t^2y'' - ty' + y = 3t^2, t > 0$$

(5 points) (a) Compute the Wronskian $W(t)$.

(10 points) (b) Find the solutions to the homogeneous problem:

$$2t^2y'' - ty' + y = 0, \ t > 0.$$

(10 points) (c) Use the method of variation of parameters to find the special solution $y_p$ to the inhomogeneous problem

$$2t^2y'' - ty' + y = 3t^2, t > 0$$

Solutions

(a) $y'' - \frac{1}{2t}y' + \frac{1}{2t^2}y = \frac{3}{2}$

$p = -\frac{1}{2t}$, $W = ce^{-\int p(t)dt} = ce^{\int \frac{1}{2t}dt} = c\sqrt{t}$

(b). Eulertype, $y = t^r$

$2r(r-1) - r + 1 = 0 \Rightarrow 2r^2 - 3r + 1 = 0 \Rightarrow r_1 = \frac{1}{2}, r_2 = 1$

$y = c_1 t^{\frac{1}{2}} + c_2 t$

(c) Let $y_p = u_1 t + u_2 t^{\frac{1}{2}}$

$u_1' t + u_2' t^{\frac{1}{2}} = 0 \quad \Rightarrow u_2' = -u_1' t^{\frac{1}{2}}$

$u_1' + \frac{1}{2}u_2' t^{-\frac{1}{2}} = \frac{3}{2}$

$u_1' - \frac{1}{2}u_1' = \frac{3}{2} \quad \Rightarrow u_1' = 3 \Rightarrow u_1 = 3t$

$u_2' = -3t^{\frac{1}{2}} \qquad u_2 = -2t^{\frac{3}{2}}$

$y_p = 3t \cdot t - 2t^{\frac{3}{2}} t^{\frac{1}{2}} = t^2$

(continuing blank page)

(25 points) 5. A mass weighing 16 lb stretches a spring 2 ft by itself.

(15 points) (a) Suppose that there is no damping and no external forces acting on the system. The spring is initially displaced 3 in upwards from its equilibrium position and given an initial velocity of 1 ft/sec downward. Find the displacement $u(t)$ at any time $t$. Write the solution in the form $R\cos(\omega_0 t - \delta)$. ($R > 0, \omega_0 > 0, 0 \le \delta < 2\pi$.)

Hint: 1 ft=12 in. $g = 32ft/sec^2$. $m = \frac{w}{g}, k = \frac{w}{l}$.

(10 points) (b) Suppose that the external force $F_e = \cos(\omega t)$ is added. Let the damping coefficient be $\gamma$. (Here $\gamma = 0$ or $\gamma > 0$.) Under what conditions on $\gamma$ and $\omega$ does the displacement remain bounded at $t \to +\infty$? State your reason.

5 (a). $m = \frac{16}{32} = \frac{1}{2}$

$k = \frac{16}{2} = 8$

$\gamma = 0$

$u(0) = 3\text{in} = -\frac{1}{4}$

$u'(0) = 1$

$m u'' + \gamma u' + k u = 0 \Rightarrow \begin{cases} \frac{1}{2}u'' + 8u = 0 \\ u(0) = -\frac{1}{4},\ u'(0) = 1 \end{cases}$ $\qquad \omega_0 = \sqrt{\frac{k}{m}} = 4$

$u = A\cos 4t + B\sin 4t$

$u = -\frac{1}{4}\cos 4t + \frac{1}{4}\sin 4t$

$A = -\frac{1}{4},\ B = \frac{1}{4} \Rightarrow R = \sqrt{(-\frac{1}{4})^2 + (\frac{1}{4})^2} = \frac{\sqrt{2}}{4}$

$\left.\begin{array}{l} \frac{\sqrt{2}}{4}\cos\delta = -\frac{1}{4} \Rightarrow \cos\delta = -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4}\sin\delta = \frac{1}{4} \Rightarrow \sin\delta = \frac{\sqrt{2}}{2} \end{array}\right\} \Rightarrow \delta = \frac{3}{4}\pi$

$u = \frac{\sqrt{2}}{4}\cos(4t - \frac{3}{4}\pi)$

(b). If either $\gamma > 0$ or $\gamma = 0, \omega \ne \omega_0$, then $u(t)$ remains bounded.

Reason: $\gamma > 0$, $y = y_h + y_p = y_h + A\cos\omega t$, $y_h \to 0$ as $t \to +\infty$

$\gamma = 0$, $y = A\cos\omega_0 t + B\sin\omega_0 t + A\cos\omega t$, if $\omega \ne \omega_0$.