AXIALLY SYMMETRIC SOLUTIONS OF ALLEN-CAHN EQUATION WITH FINITE MORSE INDEX*

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Abstract. In this paper we study axially symmetric solutions of Allen-Cahn equation with finite Morse index. It is shown that there does not exist such a solution in dimensions between 4 and 10. In dimension 3, we prove that these solutions have finitely many ends. Furthermore, the solution has exactly two ends if its Morse index equals 1.

1. Introduction

In this paper we study axially symmetric solutions of the Allen-Cahn equation

$$\Delta u = W'(u), \quad \text{in } \mathbb{R}^{n+1}. \tag{1.1}$$

Here $W(u)$ is a general double well potential, that is, $W \in C^3([-1,1])$ satisfying

- $W > 0$ in $(-1,1)$ and $W(\pm 1) = 0$;
- $W'(\pm 1) = 0$ and $W''(-1) = W''(1) = 2$;
- W is even and 0 is the unique critical point of W in $(-1,1)$.

A typical model is given by $W(u) = (1-u^2)^2/4$.

For this class of double well potential W, there exists a unique solution to the following one dimensional problem

$$g''(t) = W'(g(t)), \quad g(0) = 0 \quad \text{and} \quad \lim_{t\to\pm\infty} g(t) = \pm 1. \tag{1.2}$$

Moreover, as $t \to \pm\infty$, $g(t)$ converges exponentially to ± 1 and the following quantity is well defined

$$\sigma_0 := \int_{-\infty}^{+\infty} \left[\frac{1}{2} g'(t)^2 + W(g(t))\right] dt \in (0, +\infty).$$

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In fact, as $t \to \pm\infty$, the following expansions hold: there exists a positive constant A such that for all $|t|$ large,

$$\begin{cases} g(t) = (1 - Ae^{-\sqrt{2}|t|}) sign(t) + O(e^{-2\sqrt{2}|t|}), \\ g'(t) = \sqrt{2}Ae^{-\sqrt{2}|t|} + O(e^{-2\sqrt{2}|t|}), \\ g''(t) = -2Ae^{-\sqrt{2}|t|} + O(e^{-2\sqrt{2}|t|}). \end{cases}$$

Denote points in $\mathbb{R}^{n+1}$ by $(x_1, \cdots, x_n, z)$ and let $r := \sqrt{x_1^2 + \cdots + x_n^2}$.

Definition 1.1.
- *A function u is axially symmetric if $u(x_1, \cdots, x_n, z) = u(r, z)$.*
- *A solution of (1.1) is stable in a domain $\Omega \subset \mathbb{R}^{n+1}$ if for any $\varphi \in C_0^\infty(\Omega)$,*

$$\mathcal{Q}_\Omega(\varphi) := \int_\Omega \left[|\nabla\varphi|^2 + W''(u)\varphi^2 \right] \geq 0.$$

- *A solution of (1.1) has finite Morse index in $\mathbb{R}^{n+1}$ if*

$$\sup_{R>0} dim \{ \mathcal{X} \subset C_0^\infty(B_R(0)) : \mathcal{Q}\lfloor_{\mathcal{X}} < 0 \} < +\infty.$$

It is well known that the finite Morse index condition is equivalent to the condition of being stable outside a compact set (see, e.g., [6]).

Definition 1.2. *An axially symmetric solution of (1.1) has finitely many ends if for some $R > 0$,*

- *$u \neq 0$ in $B_R^n(0) \times \{|z| > R\}$;*
- *outside $\mathcal{C}_R := B_R^n(0) \times \mathbb{R}$, $\{u = 0\}$ consists of finitely many graphs Γ_α, where*

$$\Gamma_\alpha = \{ z = f_\alpha(r) \}, \quad \alpha = 1, \cdots, Q,$$

and $f_1 < \cdots < f_Q$.

Our first main result is

Theorem 1.3. *If $3 \leq n \leq 9$, any axially symmetric solution of (1.1), which is stable outside a cylinder $\mathcal{C}_R$, depends only on z.*

In other words, the solution has exactly one end and it is one dimensional, i. e. all of its level sets are hyperplanes of the form $\{z = t\}$. Therefore for $3 \leq n \leq 9$, there does not exist axially symmetric solutions which is stable outside a cylinder, except the trivial ones (i.e., constant solutions ± 1 and g in (1.2)).

The dimension bound in this theorem is *sharp*. On one hand, if $n \geq 10$, there do exist *stable*, axially symmetric solutions of (1.1) in $\mathbb{R}^{n+1}$ with two ends, see Agudelo-Del Pino-Wei [1]. (The two-end solutions constructed in the paper for $3 \leq n \leq 9$ are also shown to be unstable by a different argument. Our proof of Theorem 1.3 will rely on an idea of Dancer and Farina [4].) On the other hand, nontrivial axially symmetric solutions with finite Morse index in $\mathbb{R}^3$ also exist. (See del Pino-Kowalczyk-Wei [5].) However we show that

Theorem 1.4. *If $n = 2$, an axially symmetric solution of (1.1) with finite Morse index has finitely many ends. Moreover, there exists a constant C such that for any $x \in \mathbb{R}^3$ and $R > 0$,*

$$\int_{B_R(x)} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq CR^2. \tag{1.3}$$

Concerning solutions with a low Morse index we first show that

Theorem 1.5. *If $n = 2$, any axially symmetric, stable solution of (1.1) depends only on z.*

Next we prove that

Theorem 1.6. *Any axially symmetric solution of (1.1) with Morse index 1 in $\mathbb{R}^3$ has exactly two ends.*

Two end solutions in $\mathbb{R}^3$ have been studied in detail in Gui-Liu-Wei [9]. They showed that for each $k \in (\sqrt{2}, +\infty)$ there exists two-ended axially symmetric solutions whose zero level set approximately look like $\{z = k \log r\}$. Parallel to R. Schoen's result in minimal surfaces [11], one may ask the following natural question:

Conjecture: All two-ended solutions to Allen-Cahn equation in $\mathbb{R}^3$ must be axially symmetric.

We introduce some notations used in the proof of Theorems 1.3-1.6. Taking (r, z) as coordinates in the plane, after an even extension to $\{r < 0\}$, an axially symmetric function u can be viewed as a smooth function defined on $\mathbb{R}^2$. Now (1.1) is written as

$$u_{rr} + \frac{n-1}{r}u_r + u_{zz} = W'(u). \tag{1.4}$$

We use subscripts to denote differentiation, e.g. $u_z := \frac{\partial u}{\partial z}$. A nodal domain of u_z is a connected component of $\{u_z \neq 0\}$. Sometimes we will identify various objects in $\mathbb{R}^{n+1}$ with the corresponding ones in the (r, z)-plane, if they have axial symmetry.

To prove Theorems 1.3-1.6 we follow from a strategy used by the second and the third authors [17]. One of the main difficulties is the possibility of an infinite tree of nodal domains of $\frac{\partial u}{\partial z}(r, z)$. Here we explore the decaying properties of the curvature to exclude this scenario.

The remaining part of this paper is organized as follows. In Section 2 we give a curvature decay estimate on level sets of u. This curvature estimate allows us to determine the topology and geometry of ends in Section 3. In Section 4 we show that interaction between different ends is modeled by a Toda system. The case $3 \leq n \leq 9$ is analysed in Section 5, while Section 6 is devoted to the proof of the $n = 2$ case. Finally, Theorem 1.5 and Theorem 1.6 are proved in Section 7.

2. Curvature decay

In this section we establish a technical result on curvature decay of level sets of u.

Let us first recall several results on stable solutions of (1.1). By [12], given a domain $\Omega \subset \mathbb{R}^{n+1}$, the condition that $\mathcal{Q}(\varphi) \geq 0$ for all $\varphi \in C_0^\infty(\Omega)$ is equivalent to the following Sternberg-Zumbrun inequality

$$\int_\Omega |\nabla\varphi|^2|\nabla u|^2 \geq \int_\Omega \varphi^2|B(u)|^2|\nabla u|^2, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.1}$$

Here

$$|B(u)|^2 := \frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2} = |A|^2 + |\nabla_T \log|\nabla u||^2, \tag{2.2}$$

where A is the second fundamental form of the level set of u and ∇_T is the tangential derivative along the level set.

The following *Stable De Giorgi* theorem in dimension 2 is well known, see [8].

Theorem 2.1. *Suppose u is a stable solution of (1.1) in $\mathbb{R}^2$. Then u is one dimensional. In particular, $|B(u)|^2 \equiv 0$.*

Using this theorem we show

Proposition 2.2. *Suppose u is an axially symmetric solution of (1.4) in $\mathbb{R}^{n+1}$, which is stable outside a cylinder $\mathcal{C}_R$. Then for any $R_i \to +\infty$ and $z_i \in \mathbb{R}$, after passing to a subsequence, $u_i(r,z) := u(R_i + r, z_i + z)$ converges to a one dimensional solution of (1.1) in $C^2_{loc}(\mathbb{R}^2)$.*

Proof. By standard elliptic estimates we can assume u_i converge to u_∞ in $C^2_{loc}(\mathbb{R}^2)$. Passing to the limit in (1.4) we see u_∞ is a solution of (1.1) in $\mathbb{R}^2$.

Because u is axially symmetric and stable outside $\mathcal{C}_R$, there exists an axially symmetric function φ which is positive outside $\mathcal{C}_R$ such that

$$\varphi_{rr} + \frac{n-1}{r}\varphi_r + \varphi_{zz} = W''(u)\varphi, \quad \text{outside } \mathcal{C}_R.$$

Define

$$\varphi^i(r,z) := \frac{1}{\varphi(R_i, z_i)}\varphi(R_i + r, z_i + z).$$

For any $R > 0$, it satisfies

$$\varphi^i_{rr} + \frac{n-1}{R_i + r}\varphi^i_r + \varphi^i_{zz} = W''(u_i)\varphi^i, \quad \text{in } B^2_R(0).$$

By definition, $\varphi^i(0) = 1$ and $\varphi^i > 0$. Then by Harnack inequality and standard elliptic estimates, after passing to a subsequence we can take a limit $\varphi^i \to \varphi^\infty$ in $C^2_{loc}(\mathbb{R}^2)$. Here φ^∞ satisfies

$$\varphi^\infty_{rr} + \varphi^\infty_{zz} = W''(u_\infty)\varphi^\infty, \quad \varphi^\infty > 0 \quad \text{in } \mathbb{R}^2.$$

Hence u_∞ is a stable solution of (1.1) in $\mathbb{R}^2$. By Theorem 2.1, u_∞ is one dimensional.

□

Corollary 2.3. *Suppose u is an axially symmetric solution of (1.4) in $\mathbb{R}^{n+1}$, which is stable outside a cylinder $\mathcal{C}_R$. For any $b \in (0,1)$, there exists an $R(b) > 0$ such that $|\nabla u| \neq 0$ in $\{|u| < 1-b\} \setminus \mathcal{C}_{R(b)}$. Moreover, if $x \in \{|u| < 1-b\} \setminus \mathcal{C}_{R(b)}$ and $x \to \infty$,*

$$|B(u)(x)| \to 0.$$

The main technical tool we need in this paper is the following decay estimate on $|B(u)|^2$.

Theorem 2.4. *Suppose u is an axially symmetric solution of (1.4) in $\mathbb{R}^{n+1}$, which is stable outside a cylinder $\mathcal{C}_R$. For any $b \in (0,1)$, there exists a constant $C(b)$ such that in $\{|u| < 1-b\} \setminus \mathcal{C}_{R(b)}$,*

$$|B(u)(r,z)|^2 \leq C(b) r^{-2}$$

and

$$|H(u)(r,z)| \leq C(b) r^{-2} \left(\log\log r\right)^2.$$

In the above $H(u)(r,z)$ denotes the mean curvature of the level set $\{u = u(r,z)\}$ at the point (r,z). The proof of this theorem is similar to the two dimensional case in [17]. By a blow up method, it is reduced to the second order estimate established in [18]. Note that here we do not impose any condition on n, because as in the proof of Proposition 2.2, the limiting problem after blow up is essentially a two dimensional problem and then the estimate in [18] is applicable.

3. Geometry of ends

In this section u denotes an axially symmetric solution of (1.4) in $\mathbb{R}^{n+1}$, $n \geq 2$, which is stable outside a cylinder $\mathcal{C}_R$. Here and henceforth, a constant $b \in (0,1)$ will be fixed and notations in the previous section will be kept. Take a constant $R_1 > R(b)$ so that it satisfies

$$C(b) R_1^{-2} \left(\log\log R_1\right)^2 < R_1^{-1}. \tag{3.1}$$

By Theorem 2.4, $\{u = 0\} \setminus \mathcal{C}_{R_1} = \cup_\alpha \Gamma_\alpha$, where $\alpha \in \mathcal{A}$ is the index. For each α, Γ_α is a connected smooth embedded hypersurface with or without boundary. Furthermore, $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ if $\alpha \neq \beta$. Finally, since u is axially symmetric, for each $\alpha \in \mathcal{A}$, Γ_α is also axially symmetric. As a consequence, Γ_α can be viewed as a smooth curve in the (r,z) plane.

Viewing Γ_α as a smooth curve in the (r,z) plane and r as a function defined on Γ_α, we have

Lemma 3.1. *Every critical point of r in the interior of Γ_α is a strict local minima.*

Proof. Assume by the contrary, there exists a point (r_*, z_*) in the interior of one Γ_α, which is a critical point of r but not a strict local minima. By Corollary 2.3, in a neighborhood of (r_*, z_*), $\Gamma_\alpha = \{r = f_\alpha(z)\}$. By our assumptions, $f_\alpha(z_*) = r_*$, $f_\alpha'(z_*) = 0$ and $f_\alpha''(z_*) \leq 0$. Hence

$$H_{\Gamma_\alpha}(r_*, z_*) \geq \frac{1}{r_*}.$$

In view of (3.1), this is a contradiction with Theorem 2.4. □

Since Γ_α is a connected smooth curve with end points (if there are) in $\partial\mathcal{C}_{R_1}$, by this lemma we see there is no local maxima and at most one local minima of r in the interior of Γ_α. There are two cases:

Type I. Γ_α is diffeomorphic to $[0,+\infty)$ and it has exactly one end point on $\partial\mathcal{C}_{R_1}$;
Type II. Γ_α is diffeomorphic to $(-\infty,+\infty)$ and its boundary is empty.

If Γ_α is of type I, r is a strictly increasing function with respect to a parametrization of Γ_α. Hence it can be represented by the graph $\{z=f_\alpha(r)\}$, where $f_\alpha\in C^4[R_1,+\infty)$. (Higher order regularity on f_α follows by applying the implicit function theorem to u.)

If Γ_α is of type II, there exists a point (R_α,z_α), which is the unique minima of r on Γ_α. As in Type I case, $\Gamma_\alpha\setminus\{(R_\alpha,z_\alpha)\}=\Gamma_\alpha^+\cup\Gamma_\alpha^-$, where $\Gamma_\alpha^\pm$ can be represented by two graphs $\{z=f_\alpha^\pm(r)\}$. Here $f_\alpha^+>f_\alpha^-$ on $(R_\alpha,+\infty)$ and $f_\alpha^+(R_\alpha)=f_\alpha^-(R_\alpha)=z_\alpha$.

Proposition 3.2. *There exists a constant $R_2>R_1$ such that for any type II end Γ_α, it holds that $R_\alpha<R_2$.*

Proof. Assume by the contrary, there exists a sequence of type II ends Γ_k such that $R_k\to+\infty$.

By Theorem 2.4, the rescalings $\Sigma_k:=R_k^{-1}\left[\Gamma_k-(0,z_k)\right]$ have uniformly bounded curvatures and their mean curvatures converge to 0 uniformly. By standard elliptic estimates, after passing to a subsequence of k, Σ_k converges to an axially symmetric, smooth minimal hypersurface Σ_∞. Moreover, there exist two functions $f_\infty^\pm\in C^2((1,+\infty))$ such that

$$\Sigma_\infty\setminus\{(1,0)\}=\left\{(r,z):z=f_\infty^\pm(r)\right\}.$$

Hence Σ_∞ is the standard catenoid. By [13], it is unstable. (Indeed, its Morse index is exactly 1.)

On the other hand, we claim that Σ_∞ inherits the stability from u, thus arriving at a contradiction. Indeed, let $u_k(r,z):=u(R_kr,R_k(z_k+z))$. It is a solution of the singularly perturbed Allen-Cahn equation

$$\Delta u_k=R_k^2W'(u_k).$$

Since u is stable outside $\mathcal{C}_{R_1}$, u_k is stable outside $\mathcal{C}_{R_1/R_k}$. Note that Σ_k is a connected component of $\{u_k=0\}$ and it is totally located outside $\mathcal{C}_1$. Therefore we can use the method in [3] to deduce the stability of Σ_∞. There are two cases.

- Suppose there exists another connected component of $\{u_k=0\}$, denoted by $\widetilde{\Sigma}_k$, also converging to Σ_∞ in a ball $B_r(p)$ for some $r>0$ and $p\in\Sigma_\infty$. By Theorem 2.4, $\widetilde{\Sigma}_k$ enjoys the same regularity as for Σ_k. Hence by the axial symmetry of $\widetilde{\Sigma}_k$ and the uniqueness of catenoid, $\widetilde{\Sigma}_k$ converges to Σ_∞ everywhere. In this case we can construct a positive Jacobi field on Σ_∞ as in [3, Theorem 4.1], which implies the stability of Σ_∞
- Suppose there is only one such a component in a fixed neighborhood $\mathcal{N}$ of Σ_∞. Since $\Sigma_\infty\subset\{r\geq 1\}$, we can take $\mathcal{N}\subset\{r>1/2\}$. Hence u_k is stable in $\mathcal{N}$. Then for any ball $B_r(p)$ with $r>0$ and $p\in\Sigma_\infty$, there exists a constant

$C > 0$ such that

$$\int_{\mathcal{N}\cap B_r(p)} \left[\frac{1}{2R_k}|\nabla u_k|^2 + R_k W(u_k)\right] \le C.$$

Because u_k is stable in $\mathcal{N} \cap B_r(p)$, the stability of Σ_∞ follows by applying the main result of [14].

The contradiction implies that R_α is bounded and the proposition is proven. □

Now $\{u = 0\} \setminus \mathcal{C}_{R_2} = \cup_\alpha \Gamma_\alpha$, where each Γ_α is of Type I. Denote $\Gamma_\alpha \cap \{r = R_2\} = \{(R_2, z_\alpha)\}$. By Proposition 2.2, after perhaps enlarging R_2, there is a positive lower bound for $|z_\alpha - z_\beta|$, $\forall \alpha \neq \beta$. Hence we can take the index α to be integers and we will relabel indices so that $z_\alpha < z_\beta$ for any $\alpha < \beta$. Furthermore, we have $f_\alpha < f_\beta$ in $[R_2, +\infty)$ for any $\alpha < \beta$.

Define the functions

$$f_\alpha^+(r) := \frac{f_\alpha(r) + f_{\alpha+1}(r)}{2}, \quad \text{for } r \in [R_2, +\infty),$$
$$f_\alpha^-(r) := \frac{f_\alpha(r) + f_{\alpha-1}(r)}{2}, \quad \text{for } r \in [R_2, +\infty).$$

By definition, $f_\alpha^+ = f_{\alpha+1}^-$. In the above we take the convention that $f_\alpha^+(r) = +\infty$ (or $f_\alpha^-(r) = -\infty$) if there does not exist any other end lying above (respectively below) Γ_α. Let

$$\mathcal{M}_\alpha := \left\{(r, z) : f_\alpha^-(r) < z < f_\alpha^+(r),\ r > R_2\right\}.$$

The following result describes the asymptotics of f_α as $r \to +\infty$.

Lemma 3.3. *There exists a constant C such that for each α, in $[R_2, +\infty)$ we have*

$$\left\{\begin{array}{l} |f_\alpha(r) - f_\alpha(R_2)| \le C \log r \left(\log\log r\right)^2, \\ |f_\alpha'(r)| \le C r^{-1} \left(\log\log r\right)^2, \quad |f_\alpha''(r)| \le C r^{-2} \left(\log\log r\right)^2, \\ |f_\alpha^{(3)}(r)| + |f_\alpha^{(4)}(r)| \le C r^{-2} \left(\log\log r\right)^2. \end{array}\right. \tag{3.2}$$

Proof. We divide the proof into three steps.

Step 1. Denote the second fundamental form of Γ_α by A_α, the mean curvature by H_α. By the decay rate on $|A_\alpha|$ (see Theorem 2.4), there exists a constant C such that for any $r \ge R_2$,

$$|f_\alpha''(r)| \le \frac{C}{r}, \quad |f_\alpha'(r)| \le C. \tag{3.3}$$

Step 2. For any $\lambda > 0$, let $\Sigma^\lambda := \lambda \Gamma_\alpha = \{z = f_\lambda(r), r \ge \lambda R_2\}$, where $f_\lambda(r) := \lambda f_\alpha(\lambda^{-1} r)$. By Theorem 2.4, as $\lambda \to 0$, f_λ are uniformly bounded in $C^{1,1}_{loc}(0, +\infty)$. Hence after passing to a subsequence of $\lambda \to 0$, $f_\lambda \to f_0$ in $C^1_{loc}(0, +\infty)$. Here f_0 satisfies the minimal surface equation in the weak sense on $\mathbb{R}^n \setminus \{0\}$. Then it is directly verified that $f_0 \equiv 0$. Since this is independent of the choice of subsequences of $\lambda \to 0$, we obtain

$$\lim_{r \to +\infty} f_\alpha'(r) = 0. \tag{3.4}$$

Step 3. By the bound on mean curvature in Theorem 2.4, in $(R_2, +\infty)$, f_α satisfies

$$\frac{f''_\alpha(r)}{(1+|f'_\alpha(r)|^2)^{3/2}} + \frac{n-1}{r}\frac{f'_\alpha(r)}{(1+|f'_\alpha(r)|^2)^{1/2}} = O\left(r^{-2}\left(\log\log r\right)^2\right). \tag{3.5}$$

Combining this equation with (3.4), an ordinary differential equation analysis leads to the first three estimates in (3.2).

The estimate on $|f^{(3)}_\alpha(r)|$ and $|f^{(4)}_\alpha(r)|$ follows by differentiating (3.5) in r, see [18, Section 7] for details. In fact, arguing as there, we can get some improved estimates, i.e. faster decay for $|f^{(3)}_\alpha(r)|$ and $|f^{(4)}_\alpha(r)|$, but these will not be needed in this paper. □

Two corollaries follow from this lemma. First the bound on f'_α implies an area growth bound.

Corollary 3.4. *There exists a constant C such that for each Γ_α, if R is large enough,*

$$\text{Area}\left(\Gamma_\alpha \cap (\mathcal{C}_R \setminus \mathcal{C}_{R_2})\right) \le CR^n.$$

Next, by this lemma and Corollary 2.3, we obtain

Corollary 3.5. *For each Γ_α , there exists an $\overline{R}_\alpha$ such that u_z has a definite sign in the open set $\{(r,z) : r > \overline{R}_\alpha, |z - f_\alpha(r)| < 1\}$.*

The following lemma gives a growth bound of the energy localized in the domain around each end.

Lemma 3.6. *For any α, there exists a constant C_α such that*

$$\int_{\mathcal{M}_\alpha \cap \{R_2 < r < R\}} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \le C_\alpha R^n, \quad \forall R > R_2.$$

Proof. This growth bound follows from the following two estimates.

Claim 1. For any $L > 0$ and $R > 0$,

$$\int_{\{R_2 < r < R, f_\alpha(r) - L < z < f_\alpha(r) + L\}} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \le C_\alpha R^n.$$

This follows by combining the trivial bound $\frac{1}{2}|\nabla u|^2 + W(u) \le C$, co-area formula and the area growth bound in Corollary 3.4.

Claim 2. If L is sufficiently large,

$$\int_{\left\{R_2 < r < R, f_\alpha(r) + L < z < f^+_\alpha(r)\right\}} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \le C_\alpha R^n.$$

This follows from the differential inequality

$$\Delta\left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \ge c\left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \quad \text{in } \left\{R_2 < r < R, f_\alpha(r) + L < z < f^+_\alpha(r)\right\}.$$

This is possible if we have chosen L large enough so that $W''(u) \ge c$ in this domain. (Note that by Corollary 2.3, away from $\cup_\alpha \Gamma_\alpha$, u is close to ± 1.) □

4. A Toda system

In this section, keeping the notations used in the previous section, u denotes an axially symmetric solution of (1.1) in $\mathbb{R}^{n+1}$ satisfying that, for some $R_2 > 0$, it is stable outside the cylinder $\mathcal{C}_{R_2}$ and

$$\{u = 0\} \setminus \mathcal{C}_{R_2} = \cup_{\alpha \in \mathbb{Z}} \Gamma_\alpha, \quad \Gamma_\alpha := \{z = f_\alpha(r), r > R_2\},$$

where $f_\alpha \in C^4([R_2, +\infty))$ and they are increasing in α.

4.1. **Fermi coordinates.** For each α, the upward unit normal vector of Γ_α at $(r, f_\alpha(r))$ is

$$N_\alpha(r) := \frac{1}{\sqrt{1 + |f'_\alpha(r)|^2}} \left(-f'_\alpha(r)\partial_r + \partial_z\right).$$

The second fundamental form of Γ_α at $(r, f_\alpha(r))$ with respect to $N_\alpha(r)$ is denoted by $A_\alpha(r)$. The principal curvatures are

$$\begin{cases} \kappa_{\alpha,i}(r) = -\frac{1}{r}\frac{f'_\alpha}{\sqrt{1+|f'_\alpha|^2}}, & 1 \le i \le n-1, \\ \kappa_{\alpha,n}(r) = -\frac{f''_\alpha}{(1+|f'_\alpha|^2)^{3/2}}. \end{cases} \tag{4.1}$$

By Lemma 3.3, we have

$$|A_\alpha(r)| \le C r^{-3/2}, \quad \forall r \ge R_2. \tag{4.2}$$

Let (r, t) be the Fermi coordinates with respect to Γ_α, that is, for any point X lying in a neighborhood of Γ_α, take $(r, f_\alpha(r)) \in \Gamma_\alpha$ to be the nearest point to X and t be the signed distance of X to Γ_α. By Theorem 2.4, these are well defined in the open set $\{(r,t) : |t| < c_F r, r > R_2\}$ for a constant $c_F > 0$. For each t, let Γ^t_α be the smooth hypersurface where the signed distance to Γ_α equals t. The mean curvature of Γ^t_α has the form

$$\begin{aligned} H_\alpha(r,t) = \sum_{i=1}^n \frac{\kappa_{\alpha,i}(r)}{1 - t\kappa_{\alpha,i}(r)} &= H_\alpha(r) + O\left(|t||A_\alpha(r)|^2\right) \\ &= H_\alpha(r) + O\left(|t| r^{-3}\right), \end{aligned} \tag{4.3}$$

where in the last step we have used (4.2).

Denote by $\Delta_{\alpha,t}$ the Beltrami-Laplace operator with respect to the induced metric on Γ^t_α. In Fermi coordinates the Euclidean Laplace operator has the following form

$$\Delta = \Delta_{\alpha,t} - H_\alpha(r,t)\partial_t + \partial_{tt}. \tag{4.4}$$

Concerning the error between $\Delta_{\alpha,t}$ and $\Delta_{\alpha,0}$, we have (see [9, 17])

Lemma 4.1. *Suppose φ is a C^2 function of r only, then*

$$\left|\Delta_{\alpha,t}\varphi(r) - \Delta_{\alpha,0}\varphi(r)\right| \le C r^{-3/2} \left(|\varphi''(r)| + |\varphi'(r)|\right). \tag{4.5}$$

Note that here, in order to get $r^{-3/2}$ in the right hand side of (4.5), we have used Lemma 3.3 and the estimate (4.2) again.

We introduce some notations.

- For $r > R$, let $D^\pm_\alpha(r)$ be the distance of $(r, f_\alpha(r))$ to $\Gamma_{\alpha\pm1}$, respectively.

- Denote $D_\alpha(r) := \min\{D_\alpha^+(r), D_\alpha^-(r)\}$.
- $M(r) := \max_\alpha \max_{s\geq r} e^{-\sqrt{2}D_\alpha(s)}$.

By Lemma 3.3, Γ_α and $\Gamma_{\alpha+1}$ are almost parallel. Proceeding as in the proof of [17, Lemma 8.3] and [18, Lemma 9.3] we get

Lemma 4.2. *For any $r > R_2$,*

$$\begin{cases} D_\alpha^+(r) = f_{\alpha+1}(r) - f_\alpha(r) + O\left(r^{-1/6}\right), \\ D_\alpha^-(r) = f_\alpha(r) - f_{\alpha-1}(r) + O\left(r^{-1/6}\right). \end{cases}$$

4.2. **Optimal approximation.** Fix a function $\zeta \in C_0^\infty(-2,2)$ with $\zeta \equiv 1$ in $(-1,1)$, $|\zeta'| + |\zeta''| \leq 16$. For all r large, let (to ease notation, dependence on r will not be written down)

$$\bar{g}(t) = \zeta\left(8(\log r)t\right) g(t) + \left[1 - \zeta\left(8(\log r)t\right)\right] \operatorname{sgn}(t), \quad t \in (-\infty, +\infty).$$

In particular, $\bar{g} \equiv 1$ in $(16\log r, +\infty)$ and $\bar{g} \equiv -1$ in $(-\infty, -16\log r)$.

Note that $\bar{g}$ is an approximate solution to the one dimensional Allen-Cahn equation, that is,

$$\bar{g}''(t) = W'\left(\bar{g}(t)\right) + \bar{\xi}(t), \tag{4.6}$$

where $\operatorname{spt}(\bar{\xi}) \in \{8\log r < |t| < 16\log r\}$, and $|\bar{\xi}| + |\bar{\xi}'| + |\bar{\xi}''| \lesssim r^{-4}$. Here and below we use the notation $\lesssim$ to mean having an upper bound of the order of the quantity.

In the following we assume u has the same sign as $(-1)^\alpha$ between Γ_α and $\Gamma_{\alpha+1}$.

Lemma 4.3. *For any $r > R_2$ (perhaps after enlarging R_2) and $\alpha \in \mathbb{Z}$, there exists a unique $h_\alpha(r)$ such that in the Fermi coordinates with respect to Γ_α,*

$$\int_{-\infty}^{+\infty} \left[u(r,t) - g_*(r,t)\right] \bar{g}'\left(t - h_\alpha(r)\right) dt = 0,$$

where for each α, in $\mathcal{M}_\alpha$ we define

$$g_*(r,t) := g_\alpha + \sum_{\beta<\alpha}\left[g_\beta - (-1)^\beta\right] + \sum_{\beta>\alpha}\left[g_\beta + (-1)^\beta\right],$$

and in the Fermi coordinates (r,t) with respect to Γ_β,

$$g_\beta(r,t) := \bar{g}\left((-1)^\beta\left(t - h_\beta(r)\right)\right).$$

Moreover, for any $\alpha \in \mathbb{Z}$,

$$\lim_{r\to+\infty}\left(|h_\alpha(r)| + |h_\alpha'(r)| + |h_\alpha''(r)| + |h_\alpha^{(3)}(r)|\right) = 0.$$

The proof of this lemma is similar to the one for [18, Proposition 4.1], although now there may be infinitely many components. Indeed, we can define a nonlinear map on $\bigoplus_\alpha C(\Gamma_\alpha)$ as

$$F(h) := \left(\int_{-\infty}^{+\infty}\left[u(y,z) - g_*(y,z;h)\right] g_\alpha'\left(y,z;h_\alpha\right) dz\right).$$

The α component of its derivative depends only on finitely many β, i.e. it has finite width. Moreover, it is diagonally dominated and hence invertible. Then this lemma follows from the inverse function theorem.

Let g_α and g_* be as in this lemma. Define $\phi := u - g_*$. In Fermi coordinates with respect to Γ_α, the equation for ϕ reads as

$$\begin{aligned} & \Delta_{\alpha,t}\phi - H_\alpha(r,t)\partial_t\phi + \partial_{tt}\phi \\ = \ & W''(g_*)\phi + \mathcal{N}(\phi) + \mathcal{I} + (-1)^\alpha g_\alpha' \mathcal{R}_{\alpha,1} - g_\alpha'' \mathcal{R}_{\alpha,2} \\ + & \sum_{\beta\neq\alpha} \left[(-1)^\beta g_\beta' \mathcal{R}_{\beta,1} - g_\beta'' \mathcal{R}_{\beta,2}\right] - \sum_\beta \xi_\beta, \end{aligned} \tag{4.7}$$

where

$$\mathcal{N}(\phi) = W'(g_* + \phi) - W'(g_*) - W''(g_*)\phi = O\left(\phi^2\right),$$

$$\mathcal{I} = W'(g_*) - \sum_\beta W'(g_\beta),$$

while for each β, in the Fermi coordinates with respect to Γ_β,

$$\xi_\beta(r,t) = \bar{\xi}\left((-1)^\beta (t - h_\beta(r))\right),$$

$$\mathcal{R}_{\beta,1}(r,t) := H_\beta(r,t) + \Delta_{\beta,t} h_\beta(r),$$

$$\mathcal{R}_{\beta,2}(r,t) := |\nabla_{\beta,t} h_\beta(r)|^2.$$

As in [18, Lemma 4.6], because $u = 0$ on Γ_α, h_α can be controlled by ϕ in the following way.

Lemma 4.4. *For each α and $r > R_2$, we have*

$$\|h_\alpha\|_{C^{2,1/2}(r,+\infty)} \lesssim \|\phi\|_{C^{2,1/2}(\mathcal{C}_r^c)} + \max_{(r,+\infty)} e^{-\sqrt{2}D_\alpha}, \tag{4.8}$$

$$\max_\alpha \|h_\alpha'\|_{C^{1,1/2}(r,+\infty)} \lesssim \|\phi_r\|_{C^{1,1/2}(\mathcal{C}_r^c)} + r^{-1/6} M(r). \tag{4.9}$$

4.3. **Toda system.** As in [9, 17], multiplying (4.7) by g_α' and integrating in t leads to

$$H_\alpha + \Delta_{\alpha,0} h_\alpha = \frac{2A^2}{\sigma_0}\left(e^{-\sqrt{2}D_\alpha^-} - e^{-\sqrt{2}D_\alpha^+}\right) + E_\alpha, \tag{4.10}$$

where E_α is a higher order term. More precisely, we have

Lemma 4.5. *For any $r > 2R_2$,*

$$\begin{aligned} |E_\alpha(r)| \ \lesssim \ & r^{-3} + r^{-\frac{1}{2}} M\left(r - 100\log r\right) + M\left(r - 100\log r\right)^{\frac{4}{3}} \\ + \ & \max_\alpha \left\|H_\alpha + \Delta_{\alpha,0} h_\alpha\right\|^2_{C^{1/2}(r-100\log r,+\infty)} + \|\phi\|^2_{C^{2,1/2}(r-100\log r,+\infty)}. \end{aligned} \tag{4.11}$$

Here it is still useful to note that by (4.2), now we can take the upper bound on the second fundamental form to be $O\left(r^{-3/2}\right)$ when using the derivation in [18].

4.4. **Estimates on ϕ.** As in [9, 18], we have

Lemma 4.6. *There exist two constants C such that for all r large,*

$$\begin{aligned}
& \max_\alpha \|H_\alpha + \Delta_{\alpha,0} h_\alpha\|_{C^{1/2}(r,+\infty)} + \|\phi\|_{C^{2,1/2}(\mathcal{C}_r^c)} \\
\leq \ & \frac{1}{2}\left[\max_\alpha \|H_\alpha + \Delta_{\alpha,0} h_\alpha\|_{C^{1/2}(r-100\log r,+\infty)} + \|\phi\|_{C^{2,1/2}(\mathcal{C}^c_{r-100\log r})}\right] \\
+ \ & CM\left(r - 100\log r\right) + Cr^{-3}.
\end{aligned}$$

As in [9], after finitely many times of iteration using Lemma 4.6, we get a constant C such that for any $r \geq R_2$,

$$|H_\alpha(r) + \Delta_{\alpha,0} h_\alpha(r)| + \|\phi\|_{C^{2,1/2}(\mathcal{C}_r^c)} \leq C\left[r^{-3} + M\left(r - 100\log r\right)\right].$$

By [18, Proposition 10.1]), $M(r) \lesssim r^{-2}\left(\log\log r\right)^2$. Hence

$$|H_\alpha(r) + \Delta_{\alpha,0} h_\alpha(r)| + \|\phi\|_{C^{2,1/2}(\mathcal{C}_r^c)} \leq Cr^{-2}\left(\log\log r\right)^2. \tag{4.12}$$

Next by [18, Proposition 7.1], we get

$$\|\phi_r\|_{C^{1,1/2}(\mathcal{C}_r^c)} \leq Cr^{-2-1/7}. \tag{4.13}$$

In view of Lemma 4.4, we get

$$\|h_\alpha'\|_{C^{1,1/2}((r,+\infty))} \leq Cr^{-2-1/7}. \tag{4.14}$$

Substituting this into (4.10) and applying Lemma 3.3, we obtain

$$f_\alpha''(r) + \frac{n-1}{r} f_\alpha'(r) = \frac{2A^2}{\sigma_0}\left[e^{-\sqrt{2}(f_\alpha(r) - f_{\alpha-1}(r))} - e^{-\sqrt{2}(f_{\alpha+1}(r) - f_\alpha(r))}\right] + O\left(r^{-2-\frac{1}{7}}\right). \tag{4.15}$$

By [18, Proposition 8.1], we get the following stability condition.

Proposition 4.7. *For any $\eta \in C_0^\infty(R_2, +\infty)$, we have*

$$\begin{aligned}
& \frac{4\sqrt{2}A^2}{\sigma_0}\int_{R_2}^{+\infty} e^{-\sqrt{2}(f_\alpha(r) - f_{\alpha-1}(r))}\eta(r)^2 r^{n-1} dr \\
\leq \ & \int_{R_2}^{+\infty}\left[1 + Cr^{-\frac{1}{6}}\right]|\eta'(r)|^2 r^{n-1} dr + C\int_{R_2}^{+\infty}\eta(r)^2 r^{n-2-\frac{1}{8}} dr.
\end{aligned} \tag{4.16}$$

5. The case $3 \leq n \leq 9$: Proof of Theorem 1.3

In this section we keep the same setting as in the previous section, with the additional assumption that $3 \leq n \leq 9$. In order to prove Theorem 1.3, we argue by contradiction and assume there are at least two ends of u. We show this assumption leads to a contradiction if $3 \leq n \leq 9$.

Take two adjacent ends $\Gamma_{\alpha-1}$ and Γ_α. Let $v_\alpha := f_\alpha - f_{\alpha-1}$ and $V_\alpha := e^{-\sqrt{2}v_\alpha}$. By (4.15) we get a constant $\mu \in (0, 1/8)$ such that

$$v_\alpha''(r) + \frac{n-1}{r} v_\alpha'(r) \leq \frac{4A^2}{\sigma_0} e^{-\sqrt{2}v_\alpha(r)} + O\left(r^{-2-\mu}\right), \quad \text{in } (R_2, +\infty). \tag{5.1}$$

Consequently,

$$-V_\alpha''-\frac{n-1}{r}V_\alpha'\leq\frac{4\sqrt{2}A^2}{\sigma_0}V_\alpha^2-V_\alpha^{-1}\left|V_\alpha'\right|^2+O\left(r^{-2-\mu}\right)V_\alpha,\quad\text{in }(R_2,+\infty).\tag{5.2}$$

For any $q\in[1/2,2)$ and $\eta\in C_0^\infty(R_2,+\infty)$, multiplying (5.2) by $V_\alpha(r)^{2q-1}\eta(r)^2r^{n-1}$ and integrating by parts leads to

$$\begin{aligned}&2q\int_{R_2}^{+\infty}V_\alpha(r)^{2q-2}\left|V_\alpha'(r)\right|^2\eta(r)^2r^{n-1}dr\\ \leq\ &\frac{4\sqrt{2}A^2}{\sigma_0}\int_{R_2}^{+\infty}V_\alpha(r)^{2q+1}\eta(r)^2r^{n-1}dr\\ +\ &C\int_{R_2}^{+\infty}V_\alpha(r)^{2q}\left[\left|\eta'(r)\right|^2+\eta(r)\left|\eta''(r)\right|+\eta(r)^2r^{-2-\mu}\right]r^{n-1}dr.\end{aligned}\tag{5.3}$$

On the other hand, substituting $V_\alpha^q\eta$ as test function into (4.16) leads to

$$\begin{aligned}&\frac{4\sqrt{2}A^2}{\sigma_0}\int_{R_2}^{+\infty}V_\alpha(r)^{2q+1}\eta(r)^2r^{n-1}dr\\ \leq\ &q^2\left[1+CR_2^{-\frac{1}{6}}\right]\int_{R_2}^{+\infty}V_\alpha(r)^{2q-2}V_\alpha'(r)^2\eta(r)^2r^{n-1}dr\\ +\ &C\int_{R_2}^{+\infty}V_\alpha(r)^{2q}\left[\left|\eta'(r)\right|^2+\eta(r)\left|\eta''(r)\right|+\eta(r)^2r^{-2-\mu}\right]r^{n-1}dr.\end{aligned}\tag{5.4}$$

Combining (5.3) and (5.4), if R_2 is sufficiently large, we get a constant $C(q)<+\infty$ such that

$$\begin{aligned}&\int_{R_2}^{+\infty}V_\alpha(r)^{2q+1}\eta(r)^2r^{n-1}dr\\ \leq\ &C(q)\int_{R_2}^{+\infty}V_\alpha(r)^{2q}\left[\left|\eta'(r)\right|^2+\eta(r)\left|\eta''(r)\right|+\eta(r)^2r^{-2-\mu}\right]r^{n-1}dr.\end{aligned}\tag{5.5}$$

If $0\leq\eta\leq1$, following Farina [7], replacing η by η^m for some $m\gg1$ and then applying Hölder inequality to (5.5) we get

$$\int_{R_2}^{+\infty}V_\alpha(r)^{2q+1}\eta(r)^{2m}r^{n-1}dr\leq C(q)\int_{R_2}^{+\infty}\left[\left|\eta'(r)\right|^2+\left|\eta''(r)\right|+r^{-2-\mu}\right]^{2q+1}r^{n-1}dr.\tag{5.6}$$

For any $R>2R_2$, take $\eta_R\in C_0^\infty(R_2,2R)$ such that $0\leq\eta_R\leq1$, $\eta_R\equiv1$ in $(2R_2,R)$, $|\eta_R'|^2+|\eta_R''|\leq16R^{-2}$ in $(R,2R)$. Substituting η_R into (5.6), we get

$$\int_{2R_2}^{R}V_\alpha(r)^{2q+1}r^{n-1}dr\leq C+CR^{n-2(2q+1)}.\tag{5.7}$$

Since $n\leq9$, we can take $2q+1=n/2$. After letting $R\to+\infty$ in (5.7) we arrive at

$$\int_{2R_2}^{+\infty}V_\alpha(r)^{\frac{n}{2}}r^{n-1}dr\leq C.\tag{5.8}$$

As in Dancer-Farina [4], this implies that

$$\lim_{r\to+\infty} r^2 e^{-\sqrt{2}v_\alpha(r)} = 0,$$

which then leads to a contradiction by applying (5.1) exactly in the same way as in [4] (see also [16]), if $n \geq 3$.

In other words, there is only one end of u. The one dimensional symmetry of u follows by applying the main results of [10] and [15], because now we have the energy growth bound from Lemma 3.6.

6. The case $n = 2$: Proof of Theorem 1.4

In this section u denotes an axially symmetric solution of (1.1) in $\mathbb{R}^3$, which is stable outside $B^2_{R_*}(0)\times(-R_*, R_*)$. Hence there exists a positive function $\varphi \in C^2(\mathbb{R}^3)$ such that

$$\Delta\varphi = W''(u)\varphi \tag{6.1}$$

outside $B^2_{R_*}(0) \times (-R_*, R_*)$.

By a direct differentiation we see u_z satisfies the linearized equation (6.1). We will show

Lemma 6.1. *Any nodal domain of u_z is not disjoint from $B^2_{R_*}(0) \times (-R_*, R_*)$.*

Before proving this lemma, let us first present some technical results.

Keeping notations as in Section 3 and Section 4, we define for each α,

$$\mathcal{N}_\alpha := \left\{X : -\frac{3}{4}D^-_\alpha\left(\Pi_\alpha(X)\right) < d_\alpha(X) < \frac{3}{4}D^+_\alpha\left(\Pi_\alpha(X)\right)\right\},$$

where $\Pi_\alpha(X)$ is the nearest point to X on Γ_α and d_α is the signed distance to Γ_α. By Theorem 2.4 and Lemma 3.3, Π_α is well defined and smooth in the open set $\{(r,z) : |d_\alpha(r,z)| < c_F r, r > R_*\}$ after perhaps enlarging R_*.

Lemma 6.2. *For each α, there exists an $R^*_\alpha > R_*$ so that the following holds.*

- (i) *There is a connected component Ω_α of $\{u_z \neq 0\} \cap \{r > R^*_\alpha\}$, which contains $\Gamma_\alpha \cap \{r > R^*_\alpha\}$ and is contained in $\mathcal{N}_\alpha$.*
- (ii) *There exists a constant C_α such that*

$$\int_{\Omega_\alpha\cap\mathcal{C}_R} u_z^2 \leq C_\alpha R^2, \quad \forall R > R^*_\alpha. \tag{6.2}$$

Proof. (i) This follows by looking at the distance type function. Indeed, for any $(r_*, z_*) \in \Gamma_\alpha$ where r_* is large, let $\varepsilon := \max\{D^+_\alpha(r_*)^{-1}, r_*^{-1}\}$ and

$$u_\varepsilon(r,z) := u\left(r_* + \varepsilon^{-1}r, z_* + \varepsilon^{-1}z\right).$$

By Proposition 2.2,

$$\lim_{r\to+\infty} D^\pm_\alpha(r) = +\infty. \tag{6.3}$$

Hence $\varepsilon \ll 1$ if $r_* \gg 1$.

Consider the distance type function Ψ_ε, which is defined by the relation

$$u_\varepsilon = g\left(\frac{\Psi_\varepsilon}{\varepsilon}\right).$$

By the vanishing viscosity method, as $\varepsilon \to 0$, in any compact set of $\{-1 \le r \le 1, -1 \le z \le 1\}$, Ψ_ε converges uniformly to

$$\Psi_\infty(r,z) := \begin{cases} 1-z, & 1/2 \le z \le 1, \\ z, & -1/2 \le z \le 1/2 \\ -1-z, & -1 \le z \le -1/2. \end{cases}$$

Moreover, because Ψ_∞ is C^1 in $\{-1 < r < 1, -1/2 < z < 1/2\}$, Ψ_ε converges in $C^1(\{-1 < r < 1, -1/2 < z < 1/2\})$. In particular, for all ε small,

$$\frac{\partial u_\varepsilon}{\partial z} = \frac{1}{\varepsilon} g'\left(\frac{\Psi_\varepsilon}{\varepsilon}\right)\frac{\partial \Psi_\varepsilon}{\partial z} < 0, \quad \text{in } \{|r| < 1/2, -1/4 < z < 1/4\}.$$

Similarly, $\frac{\partial u_\varepsilon}{\partial z} > 0$ in $\{|r| < 1/2, -4/5 < z < -3/4\} \cup \{|r| < 1/2, 3/4 < z < 4/5\}$. Rescaling back we get the conclusion.

(ii) This follows by adding the estimates of Lemma 3.6 in $\alpha, \alpha+1$ and $\alpha - 1$. □

Lemma 6.3. *Suppose Ω is a nodal domain of u_z, which is disjoint from $B^2_{R_*}(0) \times (-R_*, R_*)$. Then*

$$\limsup_{r\to+\infty} \frac{1}{r^2}\int_{\Omega\cap B_r(0)} u_z^2 = +\infty.$$

Proof. Assume by the contrary, there exists a constant C such that for all r large,

$$\int_{\Omega\cap B_r(0)} u_z^2 \le Cr^2.$$

Then the standard Liouville type theorem applies to the degenerate equation (see [8, 2])

$$\operatorname{div}\left(\varphi^2 \nabla \frac{u_z}{\varphi}\right) = 0,$$

which implies that $u_z \equiv 0$ in Ω. This is a contradiction. □

Proof of Lemma 6.1. Assume by the contrary, there is a nodal domain of u_z disjoint from $B^2_{R_*}(0) \times (-R_*, R_*)$. Denote it by Ω and assume without loss of generality $u_z > 0$ in Ω. Since for any $R, r > 0$,

$$|\mathcal{C}_R \cap B_r(0)| \le CR^2 r,$$

Lemma 6.3 implies that Ω cannot be totally contained in $\mathcal{C}_R$. In other words, Ω is unbounded in the r direction.

Let Ω_α be defined as in Lemma 6.2. Then we claim that

Claim. There exists at most one α such that $\Omega_\alpha \subset \Omega$.

To prove this claim, we assume by the contrary that there are $\alpha \ne \beta$ such that $\Omega_\alpha \cup \Omega_\beta \subset \Omega$. Since $u_z > 0$ in $\Omega_\alpha \cup \Omega_\beta$, $|\alpha - \beta| \ge 2$. In particular, there exists a γ lying between α and β. Moreover, $u_z < 0$ in Ω_γ.

Let $\widetilde{\Omega}$ be the nodal domain of u_z containing Ω_γ. Viewing all of these domains as open sets in the (r, z) plane, Ω_α and Ω_β can be connected by a continuous curve totally contained in Ω, which together with Γ_α and Γ_β forms a simple unbounded Jordan curve. This curve divides the plane into at least two domains, $\widetilde{\Omega}$ lying on one side and $B^2_{R_*}(0) \times (-R_*, R_*)$ on the other side.

Then there are only finite many of ends of u in $\widetilde{\Omega}$, and we can add the estimates in Lemma 3.6 to arrive at

$$\int_{\widetilde{\Omega}\cap B_R(0)} |\nabla u|^2 \leq C_{\alpha\beta} R^2, \quad \forall R \quad \text{large}.$$

This is a contradiction with Lemma 6.3, which finishes the proof of the Claim.

By this Claim, there exists an $R_3 > 0$ such that $\Omega \cap \{r > R_3\} \subset \{f_{\alpha-1}(r) < z < f_{\alpha+1}(r)\}$. Using Lemma 3.6 again, we get a constant C such that

$$\int_{\Omega\cap B_R(0)} |\nabla u|^2 \leq C R^2, \quad \forall R \quad \text{large}.$$

Since Ω is assumed to be disjoint from $B^2_{R_*}(0) \times (-R_*, R_*)$, applying Lemma 6.3 again we get a contradiction. This completes the proof. $\square$

Since u is smooth, the number of connected components of $\{u_z \neq 0\} \cap B_{2R_*}(0)$ is finite. Then by the above lemma we obtain

Corollary 6.4. *There are only finitely many nodal domains of u_z.*

Now we come to the proof of Theorem 1.4.

Proof of Theorem 1.4. By the previous corollary, nodal domains of u_z are denoted by Ω^m, $m = 1, \cdots, N$ for some $N \in \mathbb{N}$.

Assume there are infinitely many ends, Γ_α. These ends are divided into N classes, $\mathcal{I}_m$ $(1 \leq m \leq N)$, that is, $\Gamma_\alpha \in \mathcal{I}_m$ if $\Omega_\alpha \subset \Omega^m$.

There is a class, say $\mathcal{I}_1$, containing infinitely many ends. Take two indicies $\alpha, \beta \in \mathcal{I}_1$ which are adjacent in $\mathcal{I}_1$. Γ_α and Γ_β are connected by a curve in Ω^1, together with Γ_α and Γ_β which gives a simple unbounded Jordan curve $\gamma_{\alpha\beta}$ in the plane. This curve divides the (r, z) plane into at least two open domains. Since u_z has the same sign in Ω_α and Ω_β, there exists a Γ_γ lying between Γ_α and Γ_β. Assume $\Omega_\gamma \subset \Omega^{M(\alpha)}$. This defines a map from $\mathcal{I}_1$ to $\{1, \cdots, N\}$. Moreover, if $\alpha, \beta \in \mathcal{I}_1$ and $\alpha \neq \beta$, then $M(\alpha) \neq M(\beta)$, in other words, $\Omega^{M(\alpha)}$ and $\Omega^{M(\beta)}$ lie on two sides of a simple Jordan curve totally contained in Ω^1. This leads to a contradiction because $\mathcal{I}_1$ is an infinite set.

Once we know that there are only finitely many ends, by Lemma 3.6 we obtain a constant C such that

$$\int_{B_R(0)\setminus \mathcal{C}_{R_*}} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq C R^2, \quad \forall R > R_*.$$

On the other hand,

$$\int_{B_R(0)\cap \mathcal{C}_{R_*}} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq C|B_R(0) \cap \mathcal{C}_{R_*}| \leq C R_*^2 R, \quad \forall R > R_*.$$

Combining these two estimates we get (1.3).

Finally, since there are only finitely many ends, by Lemma 3.3, there exist two constants $C_4, R_4 > 0$ such that $\{u = 0\} \setminus \mathcal{C}_{R_4} \subset \{|z| < C_4 r\}$. From this we see the existence of $R > 0$ such that u does not change sign in $\mathcal{C}_R \cap \{|z| > R\}$. □

7. Bound on number of ends: Proof of Theorems 1.5 and 1.6

Since the quadratic energy growth bound has been established in Theorem 1.4, the method in dimension 2 (see [17]) can be extended to our setting, which gives

Lemma 7.1. *Suppose u is an axially symmetric solution of* (1.1) *with Morse index $N \geq 1$ in $\mathbb{R}^3$. Then for any $e \in \mathbb{R}^3$, there are at most $2N$ nodal domains of $u_e := e \cdot \nabla u$.*

We first use this lemma to prove Theorem 1.5.

Proof of Theorem 1.5. If u is stable, by Lemma 7.1, u_z does not change sign. Then we can apply the main result in [2] to deduce the one dimensional symmetry of u. Furthermore, by the axial symmetry, $u(r, z) \equiv g(z - t)$ for some $t \in \mathbb{R}$. □

Concerning solutions with Morse index 1, we first show

Lemma 7.2. *An axially symmetric solution of* (1.1) *with Morse index* 1 *has at most three ends.*

Proof. If the Morse index of u is 1, by Lemma 7.1 and Theorem 1.5, there are exactly two nodal domains of u_z.

Assume there are at least 4 ends. Take 4 adjacent ones, Γ_α, $\alpha = 1, \cdots, 4$. Recall the notation Ω_α defined in Lemma 6.2. Assume $u_z > 0$ in Ω_1 and Ω_3, $u_z < 0$ in Ω_2 and Ω_4. Since $\{u_z > 0\}$ is a connected set, there is a continuous curve connecting Γ_1 and Γ_3 in $\{u_z > 0\}$, which gives a simple unbounded Jordan curve contained in $\{u_z > 0\}$. Clearly Ω_2 and Ω_4 lies on different sides of this curve, therefore $\{u_z < 0\}$ cannot be a connected set. This gives at least three nodal domains of u_z, a contradiction. □

Lemma 7.3. *Suppose u is an axially symmetric solution of* (1.1) *with Morse index* 1*. Then $u_r > 0$ or $u_r < 0$ strictly in $\{r \neq 0\}$.*

Proof. First note that $\{u_r = 0\} \subset \{u_{x_1} = 0\}$. Hence it cannot have interior points. Assume by the contrary that there exist zero points of u_r in $\{r \neq 0\}$. Then $\{u_{x_1} = 0\} \cap \{r \neq 0\} \neq \emptyset$. Because most part of $\{u_{x_1} = 0\}$ are smooth surfaces, $\{u_{x_1} > 0\} \cap \{r \neq 0\} \neq \emptyset$ and $\{u_{x_1} < 0\} \cap \{r \neq 0\} \neq \emptyset$. From this and the axial symmetry we deduce the existence of two open domains $\Omega^\pm$ in the (r, z) plane, where $u_r > 0$ in Ω^+ and $u_r < 0$ in Ω^-. Viewing them as open domains in $\mathbb{R}^3$, then $\Omega^+ \cap \{x_1 > 0\}$ and $\Omega^- \cap \{x_1 < 0\}$ are two connected components of $\{u_{x_1} > 0\}$, while $\Omega^+ \cap \{x_1 < 0\}$ and $\Omega^- \cap \{x_1 > 0\}$ are two connected components of $\{u_{x_1} < 0\}$. Hence there are at least four nodal domains of u_{x_1}, a contradiction with Lemma 7.1. □

Proof of Theorem 1.6. In view of Lemma 7.2, we only need to exclude the possibility of three ends.

By Lemma 7.3, we can assume $u_r > 0$ in $\{r \neq 0\}$. Hence each connected component Γ_α of $\{u = 0\}$ is a graph in the r-direction. There are two cases:

Type I. Γ_α is not disjoint from the z axis, hence it has the form $\{r = f_\alpha(z)\}$ where f_α is a function defined on an interval $[z_\alpha^-, z_\alpha^+)$ of the z axis and $f_\alpha(z_\alpha^-) = 0$;

Type II. Γ_α is disjoint from the z axis, hence it has the form $\{r = f_\alpha(z)\}$ where f_α is a function defined on an open interval (z_α^-, z_α^+) of the z axis.

For type I, we have $\lim_{z \to z_\alpha^+} f_i(z) = +\infty$, thus Γ_α contributes one end. For Type II, we must have $\lim_{z \to z_\alpha^\pm} f_i(z) = +\infty$, thus Γ_α contributes two ends. Since u has three ends, there are either three Type I components or one Type I plus one Type II components. Therefore u can change sign one time or three times on the z-axis.

Case 1. u changes sign three times on the z-axis.

In this case, there is an interval (a^-, a^+) such that $u(0, z) < 0$ in (a^-, a^+) and $u(a^-) = u(a^+) = 0$. Let $\{z = f^\pm(r)\}$ be the connected components of $\{u = 0\}$ emanating from $(0, a^\pm)$ respectively. Because $u_r > 0$, $f^+(r)$ is decreasing in r and f^- is increasing. Hence

$$\lim_{r \to +\infty} \left(f^+(r) - f^-(r)\right) \leq a^+ - a^-.$$

This is a contradiction with Proposition 2.2.

Case 2. u changes sign one time on the z-axis.

Without loss of generality, assume $u(0, 0) = 0$, $u(0, z) > 0$ for $z > 0$ and $u(0, z) < 0$ for $z < 0$. There exists a connected component of $\{u = 0\}$ emanating from $(0, 0)$, in the form $\{z = f(r)\}$. As in Case 1, f is decreasing in r. In particular, $u > 0$ in $\{z > 0\}$. The other component of $\{u = 0\}$ is Type II, which is represented by the graphs $\{z = f^\pm(r)\}$ for two functions $f^+ > f^-$ defined on $[R_*, +\infty)$ for some $R_* > 0$. Here f^+ is still increasing in r. As in Case 1 we get

$$\lim_{r \to +\infty} \left(f(r) - f^+(r)\right) < +\infty,$$

a contradiction with Proposition 2.2 again. □

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