NYQUIST CRITERION

PROPOSITION LET $f(z)$ BE ANALYTIC INSIDE AND ON A CLOSED COUNTERCLOCKWISE CURVE C. THEN,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N_0(f)$$

$N_0(f)$: # ZEROES OF $f(z)$ INSIDE C (COUNTING MULTIPLICITY: I.E. A ZERO OF ORDER m IS COUNTED m TIMES)

PROOF DEFINE $g(z) = f'(z)/f(z)$. THEN $g(z)$ IS ANALYTIC INSIDE C EXCEPT AT SIMPLE POLES THAT OCCUR WHEN $f(z) = 0$.

THEREFORE,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{K} \text{RES}\left[\frac{f'(z)}{f(z)}\,;\,z_j\right] \qquad K = \#\text{ ZEROES OF } f(z)$$

NOW $\text{RES}\left[\frac{f'(z)}{f(z)}\,;\,z_j\right]$ WE MUST CALCULATE.

NEAR $z = z_j$ WE HAVE $f(z) = (z-z_j)^{m_j}\,h(z)$, WHERE $h(z)$ IS ANALYTIC AND $h(z_j) \neq 0$. HERE m_j IS THE ORDER OF THE ZERO AT z_j.

NOW

$$\frac{f'(z)}{f(z)} = \frac{m_j(z-z_j)^{m_j-1}h(z) + (z-z_j)^{m_j}h'(z)}{(z-z_j)^{m_j}h(z)} = \frac{m_j}{(z-z_j)} + \frac{h'(z)}{h(z)}.$$

NOW LET $z \to z_j$, WE HAVE

$$\frac{f'(z)}{f(z)} = \frac{m_j}{(z-z_j)} + \frac{h'(z_j)}{h(z_j)}.$$

WE CALCULATE $\text{RES}\left[\frac{f'}{f}\,;\,z_j\right] = m_j$. HENCE, WE CALCULATE

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{K} m_j = N_0(f).$$

FOR SIMPLE ZEROES, $m_j = 1 \;\forall j$, AND $N_0(f) \equiv K$.

NOW WE LET $z = z(t)$ PARAMETRIZE C. THEN,

$$I = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))}{f(z(t))}\,z'(t)\,dt = \frac{1}{2\pi i}\int_a^b \frac{d}{dt}\left[\log(f(z(t)))\right]dt$$

INTEGRATING ONCE, WE GET

$$I = \frac{1}{2\pi i}\left[\log(f(z(b))) - \log(f(z(a)))\right]$$

HENCE
$$I = \frac{1}{2\pi i}\left[\Delta(\log|f(z)|) + i\,\Delta(\arg(f(z)))\right]$$
Δ = change over the circuit.

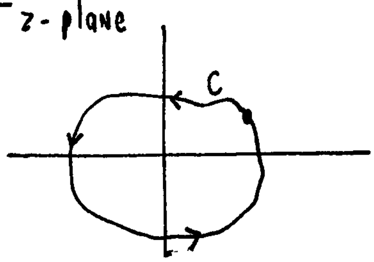
HOWEVER, $f(z(a)) = f(z(b))$ SINCE $z(a) = z(b)$.

THUS $$I = \frac{1}{2\pi}\Delta(\arg(f(z))).$$

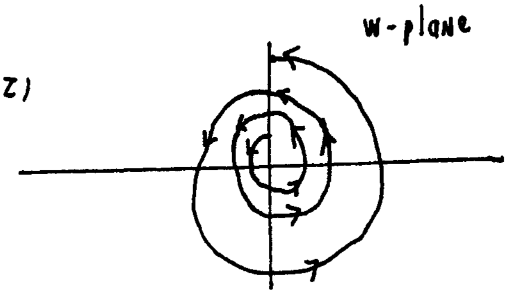
THEREFORE, WE HAVE THE FINAL RESULT,

$$\frac{1}{2\pi}\Delta(\arg(f(z))) = N_0(f).$$

EXAMPLE



$w = f(z)$



$\Delta(\arg(w)) = 6\pi$

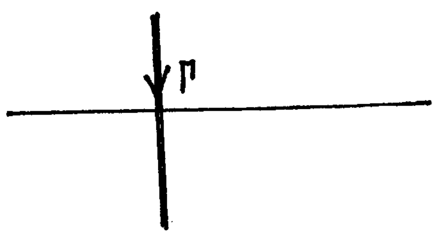
HENCE $N_0(f) = 3$ inside C.

APPLICATION (NYQUIST)

LET $f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$ a_i's ARE REAL.

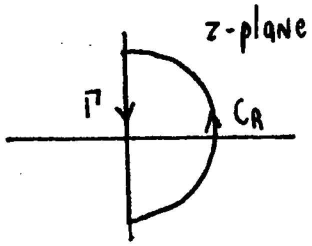
ASSUME $f(z)$ HAS NO ZEROES ON IMAGINARY AXIS. THEN, THE CLAIM IS THAT THERE ARE NO ZEROES IN RIGHT HALF-PLANE WHEN

$$\Delta_\Gamma\left[\arg(f(iy))\right] = -n\pi$$ WHERE n = degree (f).



Γ = imaginary axis directed downwards.

PROOF CONSIDER THE CONTOUR AS SHOWN BELOW



$$\lim_{R\to\infty}\left[\frac{1}{2\pi}\Delta_{C_R}\left(\arg(f(z))\right)+\frac{1}{2\pi}\Delta_{\Gamma}\left(\arg(f(z))\right)\right] = N_0(f).$$

NOW $f(z) = a_0 z^n\left[1+\frac{a_1}{a_0 z}+\frac{a_2}{a_0 z^2}+\cdots\right]$

$\arg(f(z)) \approx n\varphi$ AS $R\to\infty$.

THEREFORE, $\lim_{R\to\infty}\Delta_{C_R}\arg(f(z)) = n\pi.$

THIS GIVES

$$N_0(f) = \frac{1}{2\pi}\left[n\pi + \Delta_\Gamma(\arg(f(iy)))\right]$$ ← ARGUMENT PRINCIPLE

THEREFORE, IF $\Delta_\Gamma \arg(f(iy)) = -n\pi$, THEN $N_0(f)=0$.

EXAMPLE 1 SHOW THAT THERE ARE NO ZEROES OF $f(z) = z^3+2z^2+z+1$ LIE IN THE LEFT 1/2 PLANE.

WE HAVE $\lim_{R\to\infty}\Delta_{C_R}(\arg(f(z))) = 3\pi.$

HENCE,

$$N_0(f) = \frac{1}{2\pi}\left(3\pi + \Delta_\Gamma(\arg(f(iy)))\right).$$

WE CALCULATE $f(iy) = (iy)^3+2(iy)^2+iy+1$

WE WRITE $f(iy) = 1-2y^2+i(1-y^2)y$

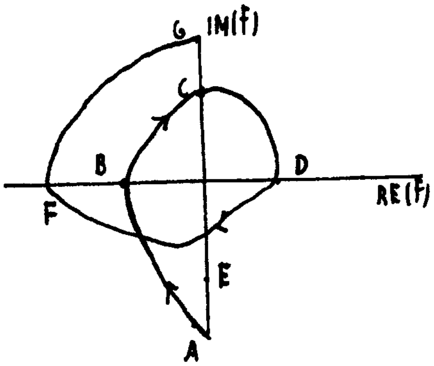
RE $f=0 \to y_{R_\pm} = \pm 1/\sqrt{2}$

IM $f=0 \to y=0,\ y_{I_\pm}=\pm 1$

	Γ	RE(f)	IM(f)
A	∞	<0	<0
B	y_{I+}	<0	=0
C	y_{R+}	=0	>0
D	0	>0	=0
E	y_{R-}	=0	<0
F	y_{I-}	<0	=0
G	$-\infty$	<0	>0

NOW AT A: $\mathrm{IM}[f]/\mathrm{RE}[f] \to \infty$

NOW AT G: $\mathrm{IM}[f]/\mathrm{RE}[f] \to -\infty$



WE GET

$\Delta_\Gamma(\arg(f(iy))) = -3\pi.$

WE THEREFORE GET

$N_0(f)=0$

⟶ NO ZEROES IN Right half-plane.

④

EXAMPLE 2 FIND THE NUMBER OF ZEROES OF

$$f(z) = z^3 - 2z^2 + 4 \quad \text{IN RIGHT HALF-PLANE.}$$

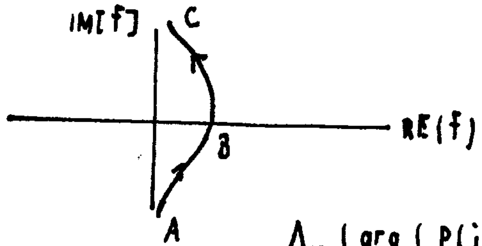
NOW $\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{C_R}(\arg(f($[redacted]$)))+\frac{1}{2\pi}\Delta_\Gamma \arg(f(iy)) = \#$ ZEROES.

NOW WE CALCULATE $\lim_{R\to\infty}\Delta_{C_R}(\arg(f(z))) = 3\pi.$

NOW $f(iy) = 2y^2 + 4 - iy^3$ $\quad$ $RE/_{IM} \to 0$ AS $y \to \pm\infty$.

$IM[f] = 0$ WHEN $y = 0$ $\quad$ $RE[f] = 2y^2+4 \quad IM[f] = -y^3.$

	y	RE(f)	IM(f)
A	∞	>0	<0
B	0	>0	$=0$
C	$-\infty$	>0	>0



$\Delta_\Gamma(\arg(P(iy))) = \pi.$

THEREFORE, $\#$ ZEROES $= \frac{1}{2\pi}(3\pi + \pi) = 2. \longrightarrow$ $\#$ ZEROES $= 2$, IN THE RIGHT HALF-PLANE.

APPLICATION CONSIDER THE DIFFERENTIAL EQUATION

$\mathcal{L}(y) = f(t)$ WITH INITIAL CONDITIONS.

WE TAKE LAPLACE TRANSFORMS TO GET

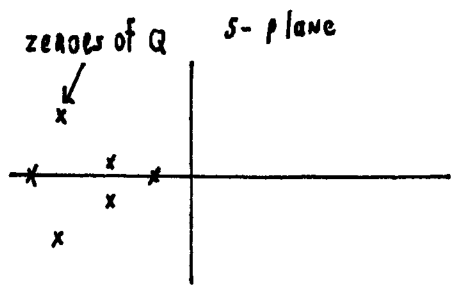
$$Y(s) = \frac{P(s)}{Q(s)}$$

WHERE, SUPPOSE, $Q(s)$ IS A POLYNOMIAL IN S, AND THAT $P(s)$ IS ANALYTIC IN S.

SUPPOSE, THAT WE CAN USE THE ARGUMENT PRINCIPLE TO SHOW THAT THERE ARE NO ZEROES OF $Q(s)$ IN THE RIGHT HALF-PLANE. THEN SINCE

$$y(t) = \sum_{j=1}^{K} RES\left[\frac{P(s)}{Q(s)}; s_j\right] e^{s_j t}$$

WE HAVE y BOUNDED AS $t \to \infty$.



EXAMPLE FIND THE NUMBER OF ZEROES OF $p(z)$ IN RIGHT-HALF-PLANE

WHERE $p(z) = z^4 + 2z^3 + 3z^2 + z + 2.$ $\lim_{R\to\infty} \Delta_{C_R} \arg(p(z)) = 4\pi$

RECALL $N_0(p) = \frac{1}{2\pi}\left[4\pi + \Delta_{\Gamma} \arg(p(iy))\right].$

NOW $p(iy) = y^4 - 2iy^3 - 3y^2 + iy + 2 = (y^4 - 3y^2 + 2) + i(y - 2y^3)$

SO $p(iy) = (y^2-1)(y^2-2) + iy(1-2y^2)$

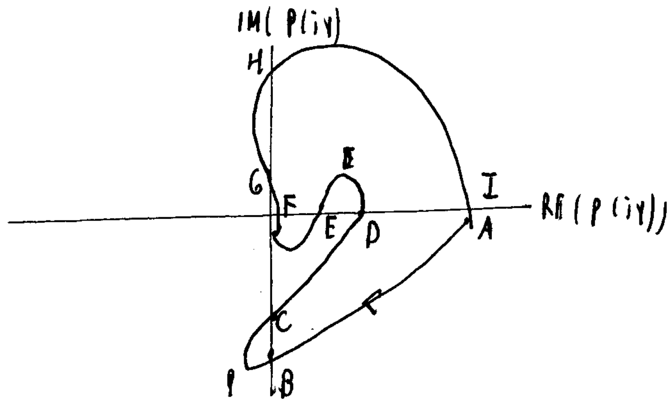
$RE[p(iy)] = 0$ when $y_{R_{1\pm}} = \pm\sqrt{2}$, $y_{R_{2\pm}} = \pm 1$

$IM[p(iy)] = 0$ when $y = 0$, $y_{I_\pm} = \pm 1/\sqrt{2}$

AS $y \to +\infty$ $RE/IM \to -\infty$ OR $IM/RE \to 0^-$

AS $y \to -\infty$, $IM/RE \to 0^+$

		$RE(p)$	$IM(p)$
A	∞	>0	<0
B	$y_{R_{1+}}$	$=0$	<0
		<0	<0
C	$y_{R_{2+}}$	$=0$	<0
D	y_{I_+}	>0	$=0$
		>0	>0
E	0	>0	$=0$
		>0	<0
F	y_{I_-}	>0	$=0$
G	$y_{R_{2-}}$	$=0$	>0
		<0	>0
H	$y_{R_{1-}}$	$=0$	>0
I	$-\infty$	>0	>0



NOTICE IT ~~IS~~ DOES NOT CIRCLE THE ORIGIN.

HENCE $\Delta_{\Gamma} \arg(p(iy)) = 0$

SO $N = \frac{1}{2\pi}(4\pi + 0) = 2$

$\longrightarrow$ There are two zeroes in RHP.