FOURIER TRANSFORMS

A COMPLEX FOURIER SERIES IS

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx .$$

PROOF $\displaystyle \int_{-L}^{L} e^{-im\pi x/L} f(x)\, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} e^{i(n-m)\pi x/L}\, dx = 2L\, c_m$ SO THAT

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-im\pi x/L}\, dx.$$

TO DERIVE FOURIER TRANSFORM,

$$f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2L}\int_{-L}^{L} f(s)\, e^{-in\pi s/L}\, ds\right) e^{in\pi x/L}$$

LET $K_n = n\pi/L$ TO GET (*) $\displaystyle f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{\pi}{L}\, \hat{F}(K_n)\, e^{iK_n x}$,

WHERE $\displaystyle \hat{F}(K_n) = \int_{-L}^{L} f(s)\, e^{-iK_n s}\, ds$. NOW $K_{n+1} - K_n = \pi/L = \Delta K$

NOW THE SUM IN (*) IS THE DISCRETIZATION IN THE K-AXIS FROM $-\infty$ TO ∞. SO

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta K\, \hat{F}(K_n)\, e^{iK_n x} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(K)\, e^{iKx}\, dK \quad \text{AS} \quad L \to \infty, \Delta K \to$$

THUS WE GET THE TRANSFORM PAIR

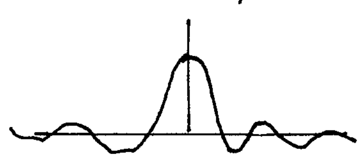
$$(+)\left\{\begin{aligned} \hat{F}(K) &= \int_{-\infty}^{\infty} f(s)\, e^{-iKs}\, ds = \mathcal{F}(f(x)) \\ f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(K)\, e^{iKx}\, dK = \mathcal{F}^{-1}[\hat{F}(K)] \end{aligned}\right.$$

FOR (+) TO BE VALID WE NEED

(i) f IS PIECEWISE-SMOOTH

(ii) $\displaystyle \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty$ SQUARE INTEGRABLE FUNCTIONS.

$\hat{F}(K)$



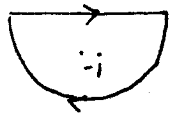
EXAMPLE 1 FIND F.T. OF $f(x) = \begin{cases} 1 & \text{IF } |x| < T \\ 0 & \text{IF } |x| > T \end{cases}$

$$\hat{F}(K) = \int_{-T}^{T} e^{-iKx}\, dx = \frac{1}{-iK} e^{-iKx}\Big|_{-T}^{T} = -\frac{i}{K}\left(e^{iKT} - e^{-iKT}\right) = \frac{2}{K}\sin(KT)$$

EXAMPLE 2 FIND F.T. OF $f(x) = \dfrac{1}{1+x^2}$.

$$\hat{F}(k) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ikx}\, dx$$

- IF $k > 0$ ENCLOSE IN LOWER 1/2 PLANE,



$\int_{C_R} \to 0$ AS $R \to \infty$

SO $$\int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ikx}\, dx + \lim_{R\to\infty} \int_{C_R} \frac{e^{-ikz}}{1+z^2}\, dz$$
$$= -2\pi i \,\mathrm{RES}\left[\frac{1}{1+x^2} e^{-ikx}; -i\right] = \pi e^{-k}$$

- IF $k < 0$ ENCLOSE IN UPPER 1/2 PLANE:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ikx}\, dx = 2\pi i \,\mathrm{RES}\left[\frac{e^{-ikx}}{1+x^2}; i\right] = \pi e^{k}$$

THUS $$\hat{F}(k) = \pi e^{-|k|}$$

PROBLEM 3 FIND F.T. OF $e^{-|x|}$.

$$\hat{F}(k) = \int_0^{\infty} e^{-x} e^{-ikx}\, dx + \int_{-\infty}^{0} e^{x-ikx}\, dx = -\frac{e^{-(x+ikx)}}{1+ik}\Big|_0^{\infty} + \frac{e^{x(1-ik)}}{1-ik}\Big|_{-\infty}^{0}$$

$$\hat{F}(k) = \frac{1}{1+ik} + \frac{1}{1-ik} = \frac{2}{1+k^2}. \longrightarrow \hat{F}(k) = \frac{2}{1+k^2}.$$

EXAMPLE 4 LET $f(x) = Ne^{-x^2/2\sigma^2}$.

THEN $$\hat{F}(k) = N\int_{-\infty}^{\infty} e^{-ikx - x^2/2\sigma^2}\, dx = N\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 + 2\sigma^2 ikx - \sigma^4 k^2)} e^{-\sigma^2 k^2/2}\, dx$$

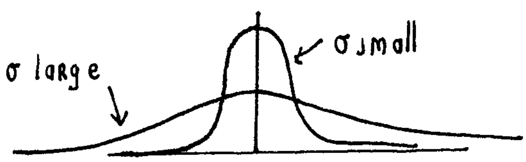
THUS, $$\hat{F}(k) = Ne^{-\sigma^2 k^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+\sigma^2 ik)^2}\, dx$$

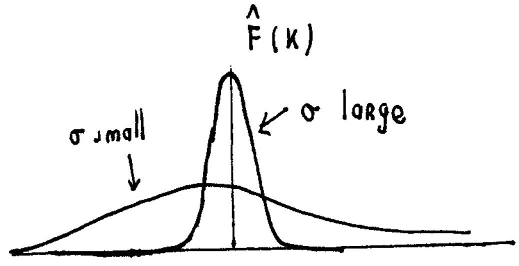
NOW BY INTEGRATING OVER A BOX IT EASILY FOLLOWS THAT $\int_{-\infty}^{\infty} e^{-(x+i\mu)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx$

THUS $$\hat{F}(k) = Ne^{-\sigma^2 k^2/2} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx.$$

RECALL THAT $\dfrac{1}{\sqrt{2\pi}\,\sigma} \displaystyle\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx = 1$. THUS, $\hat{F}(k) = \sqrt{2\pi}\,\sigma N e^{-\sigma^2 k^2/2}$

SO $f(x) = Ne^{-x^2/2\sigma^2}$





LET $N = \dfrac{1}{\sqrt{2\pi}\,\sigma}$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/\sigma^2} \qquad \hat{F}(K) = e^{-\sigma^2 K^2/2}$$

NOW WE LET $\sigma \to 0$ THEN $\int_{-\infty}^{\infty} f(x)\,dx = 1$ AND SO

$$f(x) \to \delta(x) \qquad \text{AND} \qquad \hat{F}(K) = 1$$

SO $\mathcal{F}[\delta(x)] = 1 = \int_{-\infty}^{\infty} \delta(x) e^{-iKx}\,dx$;

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iKx}\,dx.$$

<u>DELTA-SEQUENCE</u> $\quad g_\sigma(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\sigma^2}$

NOW $\lim_{\sigma \to 0} g_\sigma(x) = \begin{cases} \infty & \text{IF } x = 0 \\ 0 & \text{IF } x \neq 0 \end{cases}$ AND $\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$ FOR ALL σ.

THUS $g_\sigma(x) \to \delta(x)$ AS $\sigma \to 0$.

FOURIER TRANSFORM PROPERTIES

$$\mathcal{F}(f(x)) = \hat{F}(K) = \int_{-\infty}^{\infty} f(x) e^{-iKx}\,dx.$$

(i) $\quad \mathcal{F}(f(x+a)) = \int_{-\infty}^{\infty} f(x+a) e^{-iKx}\,dx = e^{iKa}\, \mathcal{F}(f(x))$

$$\mathcal{F}(f(x+a)) = e^{iKa}\, \mathcal{F}(f(x)).$$

<u>EX:</u> $\quad \mathcal{F}\left[\dfrac{1}{(x+1)^2+1}\right] = e^{iK}\, \mathcal{F}\left(\dfrac{1}{1+x^2}\right) = \pi e^{iK} e^{-|K|}$ SINCE $\mathcal{F}\left(\dfrac{1}{1+x^2}\right) = \pi e^{-|K|}$

(ii) $\quad \mathcal{F}[f(x/b)] = \int_{-\infty}^{\infty} f(x/b)\, e^{-iKx}\,dx = b \int_{-\infty}^{\infty} f(s)\, e^{-i(Kb)s}\,ds = b\,\hat{F}(bK).$

$b > 0 \qquad$ HENCE, $\quad \mathcal{F}[f(x/b)] = b\,\hat{F}(bK).$

<u>EX:</u> NOW $\quad \mathcal{F}\left[\dfrac{1}{\omega^2+x^2}\right] = \dfrac{1}{\omega^2}\, \mathcal{F}\left[\dfrac{1}{1+x^2/\omega^2}\right] = \dfrac{1}{\omega^2}\, \omega\, \hat{F}(\omega K) = \dfrac{1}{\omega}\, \pi e^{-\omega|K|}.$

$$\mathcal{F}\left[\frac{1}{\omega^2+x^2}\right] = \frac{\pi}{\omega}\, e^{-\omega|K|}.$$

iii) $\mathcal{F}[f'] = \int_{-\infty}^{\infty} f' e^{-ikx}\,dx = f e^{-ikx}\Big|_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx.$

THUS $\hat{\mathcal{F}}[f^{(n)}(x)] = (ik)\,\hat{\mathcal{F}}[f^{(n-1)}(x)].$

HENCE IF $f, f', \ldots, f^{(n-1)} \to 0$ AS $|x| \to \infty$

THEN $\hat{\mathcal{F}}[f^{(n)}(x)] = (ik)^n \hat{\mathcal{F}}[f(x)].$

(iv) <u>CONVOLUTION</u> $(f * g)(x) = \int_{-\infty}^{\infty} f(x-x')\,g(x')\,dx' = \int_{-\infty}^{\infty} f(x')\,g(x-x')\,dx'.$

$$\hat{\mathcal{F}}[f * g] = \int_{-\infty}^{\infty} (f*g)\, e^{-ikx}\,dx = \int_{-\infty}^{\infty} dx\left(\int_{-\infty}^{\infty} f(x-x')\,g(x')\,dx'\right) e^{-ikx}$$

$$= \int_{-\infty}^{\infty} g(x')\,e^{-ikx'}\left(\int_{-\infty}^{\infty} f(s)\,e^{-iks}\,ds\right)dx' = \left(\int_{-\infty}^{\infty} g(x')\,e^{-ikx'}\,dx'\right)\left(\int_{-\infty}^{\infty} f(s)\,e^{-iks}\,ds\right).$$

$$\hat{\mathcal{F}}[f * g] = \hat{G}(k)\,\hat{F}(k)$$

<u>EX</u> NOW $\hat{G}(k) = \pi e^{-|k|}$, $\hat{F}(k) = \dfrac{2}{1+k^2}$. SO TO INVERT

$$\hat{H}(k) = \frac{2\pi\, e^{-|k|}}{1+k^2}$$

THEN $g(x) = \dfrac{1}{1+x^2}$, $f(x) = e^{-|x|}$ SO THAT

WE GET $h(x) = \int_{-\infty}^{\infty} f(x-x')\,g(x')\,dx'.$

(v) <u>PARSEVAL'S IDENTITY</u>

$$\int_{-\infty}^{\infty} |f(x')|^2\,dx' = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{F}(k)|^2\,dk.$$

$\int_{-\infty}^{\infty} f(x-x')g(x')\,dx' = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)\hat{G}(k)e^{ikx}\,dk$ (AND SET $x=0$)

(vi) <u>COSINE TRANSFORM</u> <u>($f(x)$ EVEN)</u>

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx = \int_{-\infty}^{0} f(x)\,e^{-ikx}\,dx + \int_{0}^{\infty} f(x)\,e^{-ikx}\,dx.$$

LET $x \mapsto -x$.

THEN, $\hat{F}(k) = \int_{0}^{\infty} f(-x)\,e^{ikx}\,dx + \int_{0}^{\infty} f(x)\,e^{-ikx}\,dx.$

NOW IF $f(x) = f(-x)$ SO THAT f IS EVEN, THEN

$$\hat{F}(k) = 2\int_{0}^{\infty} f(x)\cos(kx)\,dx \longrightarrow \hat{F}(k) = \hat{F}(-k).$$

SIMILARLY $f(x) = \dfrac{1}{\pi}\int_{0}^{\infty} \hat{F}(k)\cos(kx)\,dk.$

(V) SINE TRANSFORM $f(x)$ ODD

NOW IF $f(x) = -f(-x)$ THEN

$$(*)\left\{\begin{aligned} \hat{F}(k) &= 2\int_0^\infty f(x)\sin(kx)\,dx \\ f(x) &= \frac{1}{\pi}\int_0^\infty \hat{F}(k)\sin(kx)\,dk \end{aligned}\right.$$

APPLICATIONS

EXAMPLE 1 $\quad u'' - \omega^2 u = -f(x) \quad -\infty < x < \infty$

$$u(\pm\infty) = 0 \qquad \omega > 0$$

$$\hat{\mathcal{F}}(u'') - \omega^2\,\hat{\mathcal{F}}(u) = -\hat{\mathcal{F}}(f)$$

$$(-k^2 - \omega^2)\,\hat{\mathcal{F}}(u) = -\hat{\mathcal{F}}(f) \quad \rightarrow \quad U(k) = \frac{\hat{F}(k)}{k^2+\omega^2} \qquad \hat{F}(k) = \hat{\mathcal{F}}(f).$$

NOW USE CONVOLUTION:

$$\hat{\mathcal{F}}^{-1}\left[\frac{1}{k^2+\omega^2}\right] = \frac{1}{\omega^2}\,\hat{\mathcal{F}}^{-1}\left[\frac{1}{1+k^2/\omega^2}\right] = \frac{1}{\omega}\,\frac{\omega}{2\omega}\,e^{-|x|\omega} = \frac{1}{2\omega}\,e^{-|x|\omega}$$

THUS $\quad u(x) = \dfrac{1}{2\omega}\displaystyle\int_{-\infty}^{\infty} e^{-\omega|x-s|} f(s)\,ds.$

EXAMPLE 2 $\quad U_t = D\,U_{xx} \quad -\infty < x < \infty,\ t > 0$

$$u(x,0) = f(x)$$

NOW TAKE F.T. IN x: DEFINE

$$U(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dk,$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,t)e^{ikx}\,dk$$

TAKE F.T. $\quad \left.\begin{aligned} U_t &= -Dk^2 U \\ U(k,0) &= \hat{F}(k) \end{aligned}\right\} \rightarrow U(k,t) = \hat{F}(k)\,e^{-Dk^2 t}.$

NOW $\hat{F}(k) = \hat{\mathcal{F}}(f(x))$. THEN $\quad \hat{\mathcal{F}}^{-1}\left[e^{-k^2\sigma^2/2}\right] = \dfrac{1}{\sqrt{2\pi}\,\sigma}\,e^{-x^2/2\sigma^2}$ WITH $\dfrac{\sigma^2}{2} = D$

$$\hat{\mathcal{F}}^{-1}\left[e^{-k^2 Dt}\right] = \frac{1}{2\sqrt{\pi Dt}}\,e^{-x^2/4Dt}. \quad \text{THEN,}$$

$$u(x,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4Dt} f(s)\,ds.$$

EXAMPLE 3 $\quad u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty,\ y > 0$

$$u(x,0) = h(x).$$

THEN, $\quad U(k,y) = \int_{-\infty}^{\infty} u(x,y)\, e^{-ikx}\, dx.$

THIS GIVES, $\quad U_{yy} - k^2 U = 0$

$$U(k,0) = \hat{H}(k)$$

NOW WE GET $\quad U(k,y) = \hat{H}(k)\, e^{-|k|y}.$

USE THE CONVOLUTION, $\quad \mathcal{F}^{-1}\left[e^{-|k|y} \right] = \dfrac{y}{\pi(x^2+y^2)}$

NOW $\quad u(x,y) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{y}{[(x-\xi)^2 + y^2]}\, h(\xi)\, d\xi.$

NOW LET, $\quad \lim\limits_{y\to 0} \dfrac{y}{\pi[(x-\xi)^2+y^2]} = \delta(x-\xi) \longrightarrow u(x,0) = h(x).$

EXAMPLE 4 $\quad u_t = u_{xx} \quad 0 < x < \infty,\ t > 0$

$$u(x,0) = 0, \quad u(0,t) = h(t) \qquad u, u_x \to 0 \text{ as } x \to \infty$$

SINCE $u(0,t)$ IS GIVEN WE TRY A SINE TRANSFORM:

$$U(k,t) = 2\int_0^{\infty} u(x,t)\sin(kx)\, dx \qquad u(x,t) = \frac{1}{\pi}\int_0^{\infty} U(k,t)\sin(kx)\, dk.$$

SO $\quad 2\int_0^{\infty} u_t \sin(kx)\, dx = 2\int_0^{\infty} u_{xx}\sin(kx)\, dx = 2\left[u_x \sin(kx)\Big|_0^{\infty} - k\int_0^{\infty} u_x \cos kx\, dx \right]$

BUT $u_x = 0$ AT $x = 0$ AND $u_x \to 0$ AT ∞. INTEGRATE AGAIN

$$\frac{d}{dt}\, 2\int_0^{\infty} u \sin(kx)\, dx = -2k\left[u\cos(kx)\Big|_0^{\infty} + k\int_0^{\infty} u \sin(kx)\, dx \right]$$

$$= 2k\, h(t) - k^2\left(2\int_0^{\infty} u \sin kx\, dx \right)$$

THUS $\quad U_t = -k^2 U + 2k\, h(t) \qquad U(k,0) = 0$

NOW $\quad U_t + k^2 U = 2k\, h(t)$

$$\left(e^{k^2 t}\, U \right)_t = 2k\, h(t)\, e^{k^2 t}.$$

INTEGRATE TO GET

$$e^{k^2 t}\, \bar{U} = 2k \int_0^t h(\tau) e^{k^2\tau}\, d\tau$$

$$\bar{U}(k,t) = 2k e^{-k^2 t} \int_0^t h(\tau) e^{k^2\tau}\, d\tau$$

NOW USE INVERSE TRANSFORM

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \sin(kx) \left(\int_0^t k h(\tau) e^{-k^2(t-\tau)}\, d\tau \right) dk.$$

INTERCHANGE ORDER OF INTEGRATION,

$$(*) \qquad u(x,t) = \frac{2}{\pi} \int_0^t h(\tau)\, J(x, t-\tau)\, d\tau$$

$$J(x, t-\tau) = \int_0^\infty k \sin(kx) e^{-k^2(t-\tau)}\, dk.$$

TO EVALUATE THE INTEGRAL LET

$$I = \int_0^\infty \cos(kx) e^{-k^2(t-\tau)}\, dk \qquad \text{THEN} \quad J = -\partial I/\partial x.$$

NOW $\quad I = \mathrm{RE}\left[\int_0^\infty e^{ikx - k^2(t-\tau)}\, dk \right] = \sqrt{\frac{\pi}{t-\tau}}\; e^{-x^2/4(t-\tau)}$ AFTER completing square etc.

THUS $\quad J = -\dfrac{\partial I}{\partial x} = \dfrac{x\sqrt{\pi}}{2(t-\tau)^{3/2}}\, e^{-x^2/4(t-\tau)}$

FROM $(*)$ $\quad u(x,t) = \dfrac{x}{\sqrt{\pi}} \displaystyle\int_0^t \frac{h(\tau)\, e^{-x^2/4(t-\tau)}}{(t-\tau)^{3/2}}\, d\tau$

DISPERSION RELATION

(i) $u_t = D\, u_{xx} \quad -\infty < x < \infty,\ t > 0; \quad u(x,0) = f(x)$

let $u = e^{ikx + \sigma t}$

substitute to get $\sigma = -Dk^2. \longrightarrow u = e^{ikx - Dk^2 t}$

NOW MULTIPLY BY $A(k)$ AND integrate $-\infty$ TO ∞ IN $k \longrightarrow$ superposition.

THUS GIVES $$u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx - Dk^2 t}\, dk$$

NOW $u(x,0) = f(x) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk \qquad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\, e^{-ikx'}\, dx'$

THUS $$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x') \left(\int_{-\infty}^{\infty} e^{ik(x-x') - Dk^2 t}\, dk \right) dx'$$

(ii) $u_t - c\,u_x = u_{xxx} \quad -\infty < x < \infty,\ t > 0$

$u(x,0) = f(x)$

let $u(x,t) = e^{ikx + \sigma t} \longrightarrow$ SUBSTITUTE TO GET $\sigma - cik = (ik)^3.$

THUS $\sigma = cik - ik^3. \longrightarrow$ wave propagation since σ is imaginary

THEN $$u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ik(x+c) - ik^3 t}\, dk$$

NOW $u(x,0) = f(x)$ SO $f(x) \Longrightarrow \quad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$

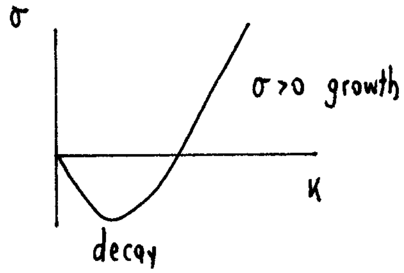
(iii) $u_t = D_0\, u_{xxxx} + D_1\, u_{xx} \quad -\infty < x < \infty$

let $u = e^{ikx + \sigma t} \longrightarrow \sigma = D_0 k^4 - D_1 k^2$

SO IF k small, large wavelength $\longrightarrow$ decay.

IF k large, small wavelength $\longrightarrow$ growth.

$D_0 > 0,\ D_1 > 0$



(iv) $u_{tt} - c^2 u_{xx} = 0$ NOW $u = e^{ikx + \sigma t} \longrightarrow \sigma^2 = -c^2 k^2. \quad \sigma = \pm ick.$

SO $$u(x,t) = \int_{-\infty}^{\infty} A_1(k)\, e^{ik(x+ct)}\, dk + \int_{-\infty}^{\infty} A_2(k)\, e^{ik(x-ct)}\, dk$$ wave propagation

MULTIPLE TRANSFORMS

$$\hat{F}(K_1, K_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, e^{-iK_1 x - iK_2 y}\, dx\, dy$$

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}(K_1, K_2)\, e^{iK_1 x + iK_2 y}\, dK_1\, dK_2 .$$

EXAMPLE

$$U_{xx} + U_{yy} - m^2 U = f(x,y)$$

$$U \longrightarrow 0 \quad \text{AS} \quad x^2 + y^2 \longrightarrow \infty$$

NOW TAKE F.T.

$$\left[+(iK_1)^2 + (iK_2)^2 \right] \bar{U} - m^2 \bar{U} = \hat{F}(K_1, K_2)$$

THUS

$$\bar{U}(K_1, K_2) = -\frac{\hat{F}(K_1, K_2)}{K_1^2 + K_2^2 + m^2}$$

NOW

$$U(x,y) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\hat{F}(K_1, K_2)}{K_1^2 + K_2^2 + m^2}\, e^{iK_1 x + iK_2 y}\, dK_1\, dK_2$$

BUT NOW

$$\hat{F}(K_1, K_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y')\, e^{-iK_1 x' - iK_2 y'}\, dx'\, dy' .$$

THEN

$$U(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y')\, G(x - x', y - y')\, dx'\, dy'$$

$$G(x,y) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{iK_1 x + iK_2 y}}{K_1^2 + K_2^2 + m^2}\, dK_1\, dK_2$$

NOW

$$G(x,y) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{iK_2 y - \sqrt{K_2^2 + m^2}\, |x|}}{\sqrt{K_2^2 + m^2}}\, dK_2$$

SINCE

$$\int_{-\infty}^{\infty} \frac{e^{iK_1 x + iK_2 y}}{K_1^2 + (K_2^2 + m^2)}\, dK_1 = \frac{e^{iK_2 y - \sqrt{K_2^2 + m^2}\, |x|}}{2i\,(K_2^2 + m^2)^{1/2}} \quad \text{BY RESIDUE CALCULUS}$$

CONSIDER $$U_{xx} - \frac{1}{c^2} U_{tt} = F(x,t) = e^{i\omega t} f(x). \qquad -\infty < x < \infty, \quad -\infty < t < \infty.$$

LET $U = e^{i\omega t} \Phi(x)$. THEN,

$$\Phi_{xx} + \frac{\omega^2}{c^2} \Phi = f(x)$$

NOW TAKE F.T. $\hat{\Phi}(k) = \hat{\mathcal{F}}[\Phi(x)]$. THEN,

$$-k^2 \hat{\Phi} + \frac{\omega^2}{c^2} \hat{\Phi} = \hat{F} \longrightarrow \hat{\Phi} = \frac{-\hat{F}(k)}{k^2 - \omega^2/c^2}$$

THEN $$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{F}(k)}{\omega^2/c^2 - k^2} e^{ikx} dk.$$ POLES ALONG REAL AXIS IN K-plane.

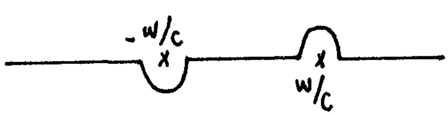
NON-UNIQUENESS SINCE WE HAVE POLES ALONG THE REAL AXIS: WHICH CONTOUR DO WE TAKE. PROBLEM IS NEED OUTGOING CONDITION AT ∞.

SUPPOSE WE WRITE
$$\hat{F}(k) = \int_{-\infty}^{\infty} f(s) e^{-iks} ds.$$

THEN, $$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) H[x-s] ds \qquad H(y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iky}}{k^2 - \omega^2/c^2} dy.$$

HOW DO WE INTERPRET $H(y)$? A PRINCIPAL VALUE INTEGRAL? NO! IT COMES FROM AN EXTRA COND
NOW TO GET OUTGOING WAVE WE WANT $H(y) = G e^{-i\omega/c |y|}$. THEN WITH $U = e^{i\omega t} \Phi$ WE GET OUTGOING WAVE. TAKE CONTOUR TO CALCULATE $H(y)$ AS SHOWN:

K-plane



$$H(y) = 2\pi i \, \mathrm{RES}\left[-\frac{1}{2\pi} \frac{e^{iky}}{k^2 - \omega^2/c^2} ; -\omega/c \right]$$ $y > 0$ enclose in upper ½ pl

$$H(y) = -2\pi i \, \mathrm{RES}\left[\frac{-1}{2\pi} \frac{e^{+iky}}{k^2 - \omega^2/c^2} ; +\omega/c \right]$$ $y < 0$ lower ½ plane enclose.

THUS $$H(y) = \frac{ic}{2\omega} e^{-i\omega/c |y|}.$$ THEN

$$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \frac{ic}{2\omega} e^{-\frac{i\omega}{c}|x-s|} ds$$

FINALLY $$U(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \frac{ic}{2\omega} e^{i\omega t - \frac{i\omega}{c}|x-s|} ds$$

TO GET AN OUTGOING WAVE WE IMPOSED A PARTICULAR PATH AROUND THE POLES AND DID NOT DO A PRINCIPAL VALUE INTEGRAL.

ANOTHER APPROACH IS TO MODIFY THE PROBLEM TO ALLOW A SMALL AMOUNT OF DAMPING

$$U_{xx} - \frac{1}{c^2} U_{tt} - \varepsilon U_t = e^{i\omega t} f(x) \qquad \varepsilon < 0$$

FOR ANY $\varepsilon > 0 \Longrightarrow U \to 0$ AS $|x| \to \infty$. SO WE WILL NOT HAVE POLES ON REAL AXIS IN F.T. PLANE.

LET $U = e^{i\omega t}\Phi(x) \longrightarrow \Phi_{xx} + \frac{\omega^2}{c^2}\Phi - \varepsilon i \omega \Phi = f(x)$

THEN, $-K^2\hat{\Phi} + \frac{\omega^2}{c^2}\hat{\Phi} - i\varepsilon\omega\hat{\Phi} = \hat{F}(K)$

THEN $\hat{\Phi}(K) = \dfrac{\hat{F}(K)}{-K^2 + \frac{\omega^2}{c^2} - i\varepsilon\omega}$. $\qquad \Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\Phi}(K)e^{iKx}\,dK$

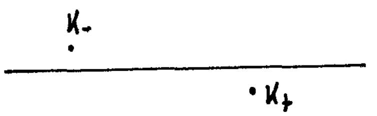
THIS LEADS TO

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, H[x-s]\, ds$$

WITH $H(y) = -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{e^{iKy}}{K^2 - \omega^2/c^2 + i\varepsilon\omega}\,dK$

poles are at $K^2 = \frac{\omega^2}{c^2}\left(1 - \frac{i\varepsilon c^2}{\omega}\right) \qquad K_\pm = \pm\frac{\omega}{c}\left(1 - \frac{i\varepsilon c^2}{2\omega}\right)$

SO $K_+ = \frac{\omega}{c} - \frac{i\varepsilon c}{2} + \ldots \qquad K_- = -\frac{\omega}{c} + \frac{i\varepsilon c}{2} + \ldots$



RE$(K_+) < 0$ RE$(K_-) > 0$. THUS FOR $\varepsilon > 0$ THE POLES ARE AS SHOWN.

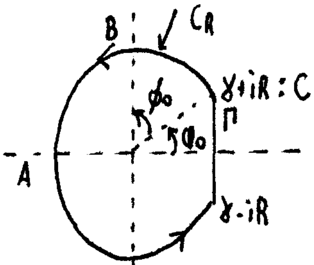
IF WE DO THE RESIDUE CALCULATION WE WILL GET SAME RESULT WHEN $\varepsilon \to 0$ AS ON PREVIOUS PAGE.

COMPLEX INTEGRATION (LAPLACE TRANSFORM)

PROPOSITION SUPPOSE $\exists\, M>0,\ K>0$ SUCH THAT ON C_R: $s = Re^{i\varphi}$ WE HAVE

$|F(s)| \le M/R^K$. THEN, $\displaystyle\lim_{R\to\infty} \left| \int_{C_R} e^{st} F(s)\, ds \right| = 0$ FOR $t>0$.

PROOF



$\phi_0 = \sin^{-1}(\gamma/R)$

CONSIDER ARC CB: $s = Re^{i\varphi}$ $\varphi_0 < \varphi < \pi/2$

$$\int_{CB} = \int_{\varphi_0}^{\pi/2} e^{Re^{i\varphi}t} F(Re^{i\varphi})\, iRe^{i\varphi}\, d\varphi$$

SO $$\left|\int_{CB}\right| \le \int_{\varphi_0}^{\pi/2} R\, e^{(R\cos\varphi)t}\, |F(Re^{i\varphi})|\, d\varphi \le \frac{M}{R^{K-1}} \int_0^{\pi/2} e^{Rt\cos\varphi}\, d\varphi$$

NOW LET $\phi = \pi/2 - \varphi$. $\cos(\varphi) = \cos(\pi/2 - \phi) = \sin\phi$.

HENCE, $$\left|\int_{CB}\right| \le + \frac{M}{R^{K-1}} \int_0^{\phi_0} e^{Rt\sin\phi}\, d\phi \le \frac{M}{R^{K-1}} \int_0^{\phi_0} e^{\gamma t}\, d\phi$$ SINCE WE

HAVE $\sin\phi \le \sin\phi_0 = \gamma/R$. THEN WE CALCULATE

$$\left|\int_{CB}\right| \le \frac{M}{R^{K-1}} \int_0^{\phi_0} e^{\gamma t}\, d\phi = \frac{M}{R^{K-1}}\, \phi_0\, e^{\gamma t} = \frac{M}{R^{K-1}} \sin^{-1}(\gamma/R)\, e^{\gamma t} \approx \frac{M\gamma}{R^K} e$$

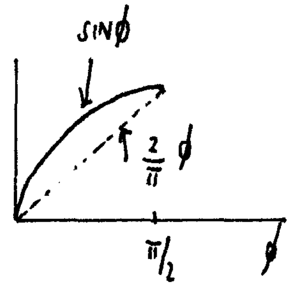
AS $R\to\infty$ ($\sin^{-1} x \approx x$ FOR $x \to 0$). THUS $\left|\int_{CB}\right| \to 0$ AS $R \to \infty$.

CONSIDER ARC AB LET $s = Re^{i\varphi}$ $-\pi/2 < \varphi < \pi$.

$$\left|\int_{AB}\right| \le \left| \int_{\pi/2}^{\pi} e^{Re^{i\varphi}t} F(Re^{i\varphi})\, iRe^{i\varphi}\, d\varphi \right| \le \frac{M}{R^{K-1}} \int_{\pi/2}^{\pi} e^{(R\cos\varphi)t}\, d\varphi = \frac{M}{R^{K-1}} \int_0^{\pi/2} e^{-(R\sin\phi)t}\, d\phi$$

SINCE $\varphi = \pi/2 + \phi$ $\cos\varphi = -\sin\phi$.

THUS $$\left|\int_{AB}\right| \le \frac{M}{R^{K-1}} \int_0^{\pi/2} e^{-(R\sin\phi)t}\, d\phi$$



BUT $\sin\phi \ge 2\phi/\pi$ ON $0 \le \phi \le \pi/2$. THUS,

$$\left|\int_{AB}\right| \le \frac{M}{R^{K-1}} \int_0^{\pi/2} e^{-2Rt\phi/\pi}\, d\phi = \frac{M}{R^{K-1}} \frac{\pi}{2Rt} \left(1 - e^{-Rt}\right) \longrightarrow 0$$

AS $R \to \infty$ IF $t > 0$.