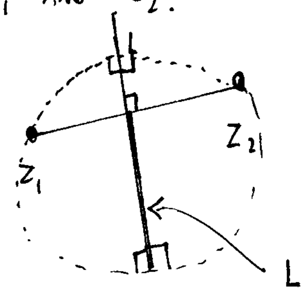
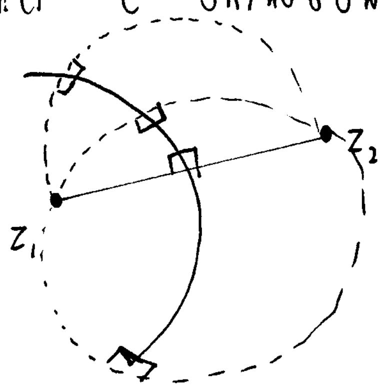
SYMMETRIC POINTS

DEFINITION TWO POINTS z_1 AND z_2 ARE CALLED SYMMETRIC POINTS WITH RESPECT TO A LINE L IF L IS THE PERPENDICULAR BISECTOR OF THE LINE JOINING z_1 AND z_2.

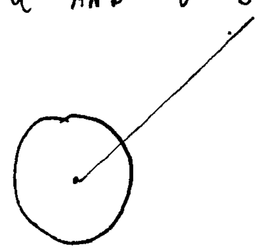


THIS IS EQUIVALENT TO EVERY LINE OR CIRCLE THROUGH z_1 AND z_2 INTERSECTS L ORTHOGONALLY

DEFINITION TWO POINTS z_1 AND z_2 ARE CALLED SYMMETRIC POINTS WRT A CIRCLE C IF EVERY STRAIGHT LINE OR CIRCLE THROUGH z_1 AND z_2 INTERSECT C ORTHOGONALLY.



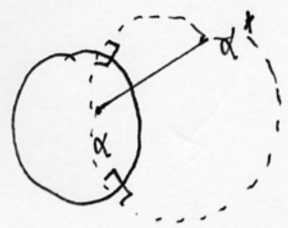
REMARK THE CENTER a OF C AND THE POINT AT ∞ ARE SYMMETRIC WRT C SINCE THERE ARE NO CIRCLES CONTAINING a AND ∞ SO THE CONDITION HOLDS.



THEOREM (SYMMETRY CONDITION) LET C BE A LINE OR A CIRCLE IN Z-PLANE AND LET $w = f(z)$ BE A BILINEAR MAP. THEN THE POINTS z_1 AND z_2 ARE SYMMETRIC WRT C IF AND ONLY IF THEIR IMAGES $w_1 = f(z_1)$ AND $w_2 = f(z_2)$ ARE SYMMETRIC WRT THE IMAGE OF C UNDER $f(z)$.

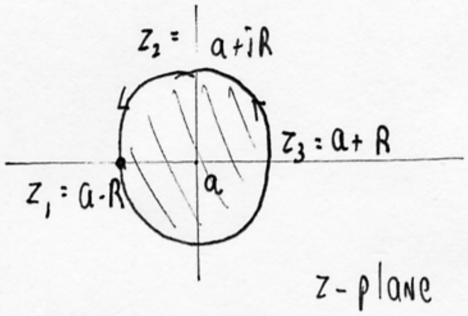
<u>PROOF</u> WE ASSUME Z_1 AND Z_2 ARE SYMMETRIC WRT A CIRCLE C. HENCE ALL LINES AND CIRCLES THROUGH Z_1 AND Z_2 INTERSECT C ORTHOGONALLY. SUPPOSE F ANALYTIC ON C AND IS MOBIUS (BILINEAR). THEN SINCE F MAPS CIRCLES AND LINES TO CIRCLES AND LINES THE ORTHOGONALITY CONDITION HOLDS BY CONFORMALITY (ANGLE PRESERVING) HENCE $W_1 = f(Z_1)$ AND $W_2 = f(Z_2)$ ARE SYMMETRIC POINTS WRT IMAGE OF C. (IF f NOT ANALYTIC ON C THEN C MAPS TO A LINE AND W_1, W_2 ARE SYMMETRIC WRT LINE). ∎

SUPPOSE WE HAVE A CIRCLE C WITH CENTER a AND RADIUS R. GIVEN A POINT α INSIDE CIRCLE, FIND A POINT α^* SYMMETRIC TO α WRT C. TO DO SO WE:



(i) MAP CIRCLE TO REAL AXIS WHERE SYMMETRIC POINT CONDITION IS SIMPLY $f(\alpha) = \overline{f(\alpha^*)}$ f BILINEAR

(ii) USE THEOREM TO CONCLUDE α, α^* ARE SYMMETRIC WRT C.

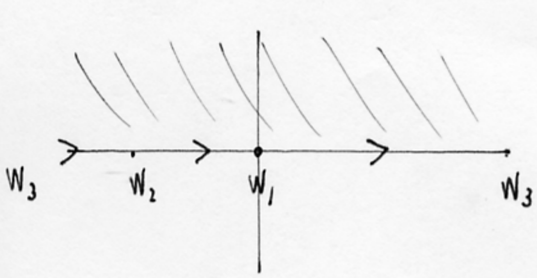


MAP $Z_1 = a - R \longrightarrow W_1 = 0$

$Z_2 = a + iR \rightarrow W_2 = -1$

$Z_3 = a + R \rightarrow W_3 = \infty$

BY ORIENTATION PRESERVING THE INSIDE OF CIRCLE BECOMES UPPER ½ PLANE



SO $W = \mathcal{B}\left(\dfrac{Z - Z_1}{Z - Z_3}\right)$

(SYM.

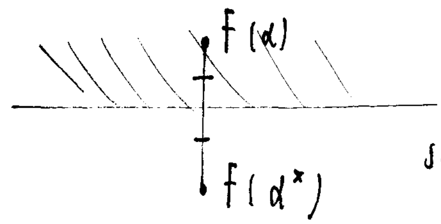
NOW $z_2 = a + iR \to w_2 = -1 \Rightarrow -1 = \beta\left(\dfrac{R(i+1)}{R(-1+i)}\right) = \beta\,\dfrac{e^{i\pi/4}}{e^{3i\pi/4}} = \beta e^{-i\pi/2} = -i$

SO $1 = i\beta$ OR $\beta = -i$

HENCE $w = f(z) = -i\left(\dfrac{z - z_1}{z - z_3}\right)$

NOW FOR $f(\alpha), f(\alpha^*)$ TO BE SYMMETRIC POINTS WRT REAL AXIS WE REQUIRE THAT

$$f(\alpha) = \overline{f(\alpha^*)}$$



SO $f(\alpha) = \overline{f(\alpha^*)}$

NOW THIS WILL MEAN THAT α, α^* ARE SYMMETRIC WRT $|z - a| = R$.
(BY THE THEOREM)

$$f(\alpha) = \overline{f(\alpha^*)} \qquad \longrightarrow \qquad -i\left(\frac{\alpha^* - z_1}{\alpha^* - z_3}\right) = i\left(\frac{\bar\alpha - \bar z_1}{\bar\alpha - \bar z_3}\right)$$
$$f(\alpha^*) = \overline{f(\alpha)}$$

SO $(\alpha^* - z_1)(\bar\alpha - \bar z_3) = -(\alpha^* - z_3)(\bar\alpha - \bar z_1)$

$$\alpha^*((\bar\alpha - \bar z_3)) - z_1(\bar\alpha - \bar z_3) = -\alpha^*((\bar\alpha - \bar z_1)) + z_3(\bar\alpha - \bar z_1$$

$$(+) \quad \alpha^*\left(2\bar\alpha - \bar z_1 - \bar z_3\right) = \bar\alpha\left(z_3 + z_1\right) - z_1\bar z_3 - z_3\bar z_1.$$

NOW $z_1 = a - R$, $z_3 = a + R$ SO $\bar z_1 + \bar z_3 = 2\bar a$ AND $z_1 + z_3 = 2a$.

$z_1\bar z_3 = (a - R)(\bar a + R) = a\bar a + aR - \bar a R - R^2$

$z_3\bar z_1 = a\bar a + \bar a R - aR - R^2$

SO $z_1\bar z_3 + z_3\bar z_1 = 2a\bar a - 2R^2$

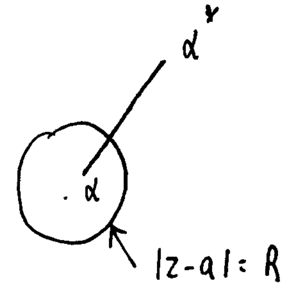
HENCE, $\alpha^*\left(2\bar\alpha - 2\bar a\right) = \bar\alpha\, 2a - 2a\bar a + 2R^2$ FROM (+)

$\Rightarrow \alpha^*(\bar\alpha - \bar a) = a(\bar\alpha - \bar a) + R^2$ SO $\boxed{\alpha^* = a + \dfrac{R^2}{(\bar\alpha - \bar a)}}$

<u>MAIN RESULT</u> α AND α^* ARE SYMMETRIC WRT $|z-a| = R$

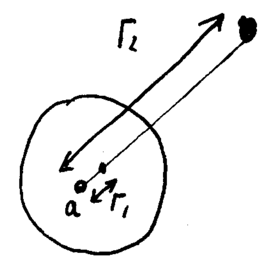
ONLY WHEN α^* AND α ARE RELATED BY

$$\alpha^* = a + \frac{R^2}{(\bar{\alpha} - \bar{a})}.$$



<u>INTERPRETATION</u> LET $\alpha - a = te^{i\varphi}$. THEN

$$\alpha^* = a + \frac{R^2}{te^{-i\varphi}} = a + \frac{R^2}{t} e^{i\varphi}.$$



SO a, α, α^* LIE ON SAME LINE (COLINEAR).

ALSO $|\alpha^* - a| \, |\bar{\alpha} - \bar{a}| = R^2$

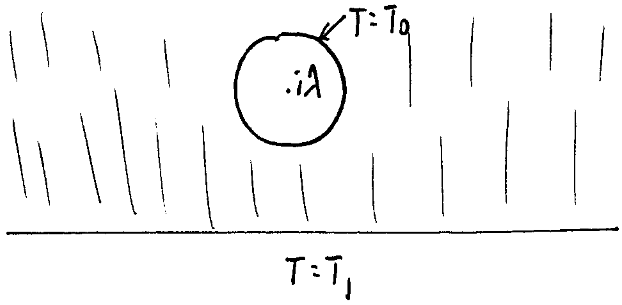
OR $|\alpha^* - a| \, |\alpha - a| = R^2. \rightarrow r_1 r_2 = R^2$

NOTICE IF $\alpha \rightarrow a$ THEN $\alpha^* \rightarrow \infty$.

<u>REMARK</u> : SUPPOSE $\alpha = 0$ AND $\alpha^* = \infty$ ARE SYMMETRIC POINTS

THEN WE MUST HAVE THEM SYMMETRIC TO A CIRCLE CENTERED AT 0.

EXAMPLE 1 SOLVE FOR $T_{xx} + T_{yy} = 0$ IN DOMAIN AS SHOWN WITH $T = T_0$ ON CIRCLE AND $T = T_1$ ON REAL AXIS.



CIRCLE $|z - i\lambda| = R$

NEED $\lambda > R$ FOR PICTURE AS SHOWN.

THERE ARE SEVERAL STEPS

STEP (i) LET z_1 BE INSIDE CIRCLE. FIND A POINT z_2 SO THAT z_2 IS SYMMETRIC WRT CIRCLE AND X-AXIS SIMULTANEOUSLY.

$\longrightarrow$ TWO EQUATIONS IN TWO UNKNOWNS FOR z_1 AND z_2.

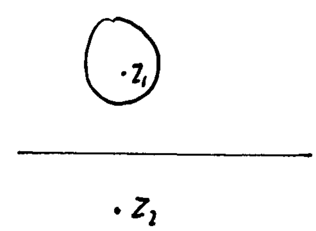
STEP (ii) MAP z_1 TO $w = 0$ AND z_2 TO $w = \infty$.

THEN $w = 0, \infty$ ARE SYMMETRIC POINTS (SINCE z_1, z_2 SYMMETRIC POINTS), BY THE THEOREM.

STEP (iii) THUS IMAGE OF REAL AXIS MUST BE CIRCLE CENTERED AT $w = 0$ AND IMAGE OF $|z - i\lambda| = R$ IS ALSO A CIRCLE CENTERED AT $w = 0$.

STEP (iv) HENCE IN W-plane we have concentric circular cylindrical geometry where it is easy to solve Laplace's EQUATION

STEP 1 LET z_1 BE INSIDE $|z - i\lambda| = R$ THEN



$$z_2 = \bar{z}_1 \longrightarrow \text{symmetric WRT X-AXIS}$$

$$z_2 = i\lambda + \frac{R^2}{\bar{z}_1 + i\lambda} \rightarrow \text{symmetric WRT circle}$$

COMBINING $(z_2 - i\lambda)(\bar{\bar{z}}_2 + i\lambda) = R^2 \longrightarrow z_2^2 = R^2 - \lambda^2$ BUT $R < \lambda$.

SO $z_2 = -i(\lambda^2 - R^2)^{1/2}$, $z_1 = i(\lambda^2 - R^2)^{1/2}$. SYMMETRIC WRT X-AXIS AND $|z - i\lambda| = R$ SIMULTANEO

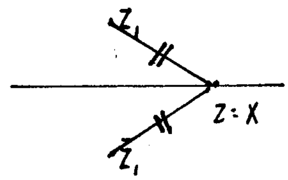
STEP 2,3 $\quad z_1 \rightarrow w = 0$

$z_2 \rightarrow w = \infty$

SO $\quad w = \dfrac{z - z_1}{z - z_2} \implies$ REAL AXIS AND $|z - i\lambda| = R$ ARE MAPPED TO CONCENTRIC CIRCLES CENTERED AT $w = 0$.

THEN NOTICE THAT SINCE $z_2 = \bar{z}_1$ WE HAVE

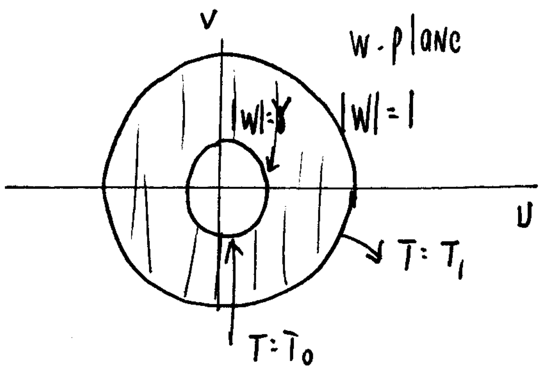
$w = \dfrac{z - z_1}{z - \bar{z}_1}$ SO FOR $z = x$ $\quad |w| = \left| \dfrac{x - z_1}{x - \bar{z}_1} \right| = \dfrac{|x - z_1|}{|x - \bar{z}_1|} = 1.$



HENCE REAL AXIS IS MAPPED TO $|w| = 1$.

NOW LET $z = i(\lambda - R)$ $\quad |w| = \left| \dfrac{i(\lambda - R) - i(\lambda^2 - R^2)^{1/2}}{i(\lambda - R) + i(\lambda^2 - R^2)^{1/2}} \right| = \left| \dfrac{(\lambda - R) - (\lambda^2 - R^2)^{1/2}}{(\lambda - R) + (\lambda^2 - R^2)^{1/2}} \right| =$

HENCE WE HAVE THE GEOMETRY



$T_{uu} + T_{vv} = 0$ IN $\gamma < (u^2 + v^2)^{1/2} < 1$

$T = T_0$ ON $(u^2 + v^2)^{1/2} = \gamma$

$T = T_1$ ON $(u^2 + v^2)^{1/2} = 1$

NOW IN POLAR COORDINATES $\quad \rho = (u^2 + v^2)^{1/2}, \quad \phi = \tan^{-1}(v/u)$

WE HAVE $\quad T_{\rho\rho} + \dfrac{1}{\rho} T_\rho + \dfrac{1}{\rho^2} T_{\phi\phi} = 0$ IN $\gamma < \rho < 1$ $\qquad \rho = |w|$

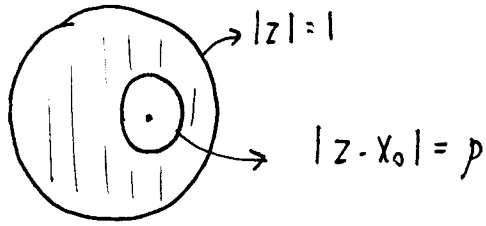
$T = T_0$ ON $\rho = \gamma$; $\quad T = T_1$ ON $\rho = 1$

SO $\quad T = A + B \log \rho$ OR WE CAN CONVENIENTLY WRITE AS

$T = T_0 + (T_1 - T_0) \log(\rho/\gamma) / \log(1/\gamma)$ $\qquad$ BUT $\rho = |w|$

SO $\quad T = T_0 + \dfrac{(T_1 - T_0)}{\log(1/\gamma)} \log \left(\dfrac{|z - z_1|}{\gamma |z - \bar{z}_1|} \right)$ IS THE SOLUTION.

EXAMPLE 2 SOLVE $T_{xx} + T_{yy} = 0$ INSIDE THE REGION AS SHOWN (SYM 7)



WITH $T = T_0$ ON BIG CIRCLE $|z|=1$

$T = T_1$ ON SMALL CIRCLE $|z-x_0| = p$

WE ASSUME $x_0 > 0$ REAL AND $x_0 + p < 1$ SO THAT SMALL CIRCLE IS INSIDE $|z|=1$.

RECALL $\alpha^* = a + R^2/(\bar{\alpha} - \bar{a})$ FORMULA.

LET z_1 BE INSIDE SMALL CIRCLE.

$$z_2 = x_0 + p^2/(\bar{z}_1 - x_0) \qquad \text{symmetric point wrt small circle}$$

$$z_2 = 0 + 1/\bar{z}_1 \qquad \text{symmetric wrt } |z|=1 \text{ circle.}$$

TWO EQUATIONS IN TWO UNKNOWNS FOR z_1 AND z_2.

SO $$z_2 = x_0 + \frac{p^2}{\left(\frac{1}{z_2} - x_0\right)} \Rightarrow (z_2 - x_0)\left(\frac{1}{z_2} - x_0\right) = p^2$$

SO $$(z_2 - x_0)(1 - x_0 z_2) = p^2 z_2 \rightarrow (z_2 - x_0)(x_0 z_2 - 1) + p^2 z_2 = 0$$

SO $$x_0 z_2^2 - z_2(1 + x_0^2 - p^2) + x_0 = 0$$

SO $$x_0 z_2^2 + z_2(p^2 - 1 - x_0^2) + x_0 = 0.$$

THIS MEANS $(z_2 - w_1)(z_2 - w_2) = 0$ SO $w_1 w_2 = 1$, $w_1 + w_2 = -\frac{1}{x_0}(p^2 - 1 -$

THIS MEANS ONE ROOT IS OUTSIDE $|z|=1$ WHICH IS THE ONE WE WANT FOR z_2.

THIS GIVES $$z_2 = -\frac{(p^2 - 1 - x_0^2)}{2x_0} + \frac{\sqrt{(p^2 - 1 - x_0^2)^2 - 4x_0^2}}{2x_0}$$

(WE TAKE + ROOT)

THE ROOT IS REAL WHEN $(p^2 - 1 - x_0^2)^2 - 4x_0^2 > 0$.

WE NOW SHOW THAT THIS IS ALWAYS TRUE WHEN $p < 1 - x_0$

WE REWRITE THIS AS

$$\left(\rho^2 - (X_0^2 - 2X_0 + 1) - 2X_0\right)^2 - 4X_0^2$$

$$= \left(\left(\rho^2 - (X_0-1)^2\right) - 2X_0\right)^2 - 4X_0^2$$

$$= \left(\rho^2 - (X_0-1)^2\right)^2 - 4X_0\left(\rho^2 - (X_0-1)^2\right) + 4X_0^2 - 4X_0^2$$

$$= \left(\rho^2 - (X_0-1)^2\right)^2 - 4X_0\left(\rho^2 - (X_0-1)^2\right)$$

BUT SINCE $\rho < 1 - X_0 \longrightarrow \rho^2 - (1 - X_0)^2 < 0$

WHICH IMPLIES THAT

$$\left(\rho^2 - 1 - X_0^2\right)^2 - 4X_0^2 = \left(\rho^2 - (X_0-1)^2\right)^2 - 4X_0\left(\rho^2 - (X_0-1)^2\right) > 0.$$

HENCE WE CONCLUDE THAT Z_2 IS <u>REAL</u> WITH $|Z_2| > 1$

WHERE

$$Z_2 = -\frac{(\rho^2 - 1 - X_0^2)}{2X_0} + \frac{1}{2X_0}\left[(\rho^2 - 1 - X_0^2)^2 - 4X_0^2\right]^{1/2},$$

AND $Z_1 = 1/\bar{Z}_2 = 1/Z_2$

NOW WE FOLLOW STEPS 2 AND 3

MAP $Z = Z_1 \rightarrow W = 0$

$Z = Z_2 \rightarrow W = \infty$

SO $W = B\left(\dfrac{Z - Z_1}{Z - Z_2}\right)$ THIS WILL MAP SMALL

CIRCLE AND $|Z| = 1$ TO CONCENTRIC CIRCLES IN W-PLANE.

CHOOSE B SO THAT $|z|=1$ MAPS TO $|w|=1$.

HENCE,
$$w = B\left(\frac{z-z_1}{z-1/\bar{z}_1}\right) = B\bar{z}_1\left(\frac{z-z_1}{z_1 z-1}\right)$$

ON $|z|=1$ WE WRITE $z=e^{i\varphi}$ AND RECALL z_1 IS REAL.

SO
$$|w| = |B||z_1|\left|\frac{e^{i\varphi}-z_1}{z_1e^{i\varphi}-1}\right|$$
$$= |B||z_1|\left(\frac{(e^{i\varphi}-z_1)(e^{-i\varphi}-z_1)}{(z_1e^{i\varphi}-1)(z_1e^{-i\varphi}-1)}\right)^{1/2}$$
$$= |B||z_1|\left(\frac{1+z_1^2-z_1e^{i\varphi}-z_1e^{-i\varphi}}{z_1^2+1-z_1e^{i\varphi}-z_1e^{-i\varphi}}\right)^{1/2} = |Bz_1|$$

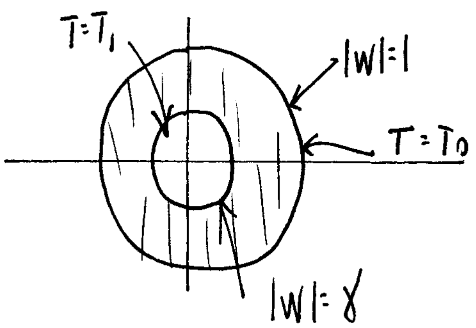
SO FOR $|w|=1 \rightarrow B = 1/z_1$.

HENCE $w = \dfrac{z-z_1}{z_1 z-1}$ MAPS $|z-x_0|=\rho$ AND $|z|=1$ TO CONCENTRIC CIRCLES

WITH $|z|=1 \longrightarrow |w|=1$.

TAKING AND POINT ON $|z-x_0|=\rho$ SUCH AS $z=x_0+\rho$ WE CALCULATE
$$|w| = \gamma = \left|\frac{x_0+\rho-z_1}{z_1(x_0+\rho)-1}\right| < 1.$$

HENCE WE HAVE

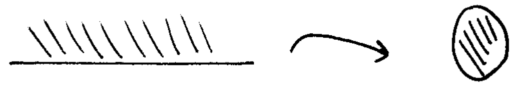


SOLUTION IS
$$T = T_0 + (T_1-T_0)\frac{\log|w|}{\log\gamma}$$

WITH $w = \dfrac{z-z_1}{z_1 z-1}$

AND z_1 GIVEN EARLIER.

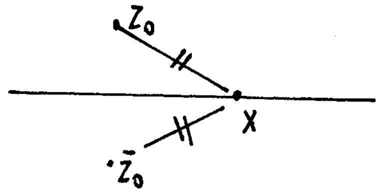
EXAMPLE 3 PROVE THAT EVERY BILINEAR MAP THAT TAKES THE UPPER HALF-PLANE TO THE INTERIOR OF THE UNIT DISK $|W|=1$ MUST HAVE THE FORM



$$W = e^{i\varphi}\left(\frac{Z-Z_0}{Z-\bar{Z}_0}\right) \qquad \text{IM } Z_0 > 0 \ , \ \varphi \text{ REAL.}$$

PROOF. LET $Z = X$ REAL THEN $|W| = \left|\dfrac{X-Z_0}{X-\bar{Z}_0}\right| = 1$ CLEARLY

SINCE WE HAVE



ALSO $Z = Z_0 \longrightarrow W=0$ INSIDE DISK

(SYMMETRIC)

NOW CHOOSE $\text{IM}(Z_0) > 0$. THEN INVERSE POINT WRT REAL AXIS IS SIMPLY $\bar{Z}_0$. RECALL SYMMETRIC POINTS ARE MAPPED TO SYMMETRIC POINTS UNDER BILINEAR MAP.

$$Z = Z_0 \longrightarrow W = 0 \qquad\qquad Z = \bar{Z}_0 \longrightarrow W = \infty$$

SO IMAGE MUST BE A CIRCLE CENTERED AT ORIGIN

$$W = B\left(\frac{Z-Z_0}{Z-\bar{Z}_0}\right)$$

NEED $|B| = 1$ FOR $|W| = 1$ TO BE IMAGE OF $Z = X$

HENCE $B = e^{i\varphi}$

EXAMPLE 4 PROVE THAT EVERY BILINEAR MAP THAT TAKES $|Z| \le 1$ INTO $|W| \le 1$ MUST HAVE THE FORM

$$W = B\left(\frac{Z-\alpha}{\bar{\alpha}Z-1}\right) \qquad \text{WITH } |B| = 1 \text{ AND } |\alpha| < 1$$

IF W ANALYTIC IN $|Z| \le 1$.

PROOF (i) LET $z = e^{i\varphi}$ SO $|z| = 1$

$$w = \beta\left(\frac{e^{i\varphi} - \alpha}{e^{i\varphi}\bar{\alpha} - 1}\right) = \beta e^{-i\varphi}\left(\frac{1 - \alpha e^{-i\varphi}}{-1 - \bar{\alpha} e^{i\varphi}}\right) = \beta e^{-i\varphi}\left(\frac{\gamma}{-\bar{\gamma}}\right) \qquad \gamma = 1 - \alpha e^{-i\varphi}$$

SO $|w| = |\beta||e^{-i\varphi}|\left|\frac{\gamma}{-\bar{\gamma}}\right| = |\beta|$ BUT SINCE $|\beta| = 1 \longrightarrow |w| = 1$

SO $|z| = 1$ MAPS TO $|w| = 1$.

AND $z = \alpha$ WITH $|\alpha| < 1 \to w = 0$ $\Longrightarrow$ $|z| \leq 1$ MAPS TO $|w| \leq 1$.

(ii) LET α WITH $|\alpha| < 1$ BE INSIDE $|z| = 1$.

THE INVERSE POINT IS $\alpha^* = 1/\bar{\alpha}$ (RECALL $\alpha^* = a + \frac{R^2}{\bar{\alpha} - \bar{a}}$)

$z = \alpha \longrightarrow w = 0$

$z = \alpha^* \longrightarrow w = \infty$

SO $$w = \gamma\left(\frac{z - \alpha}{z - \alpha^*}\right) = \gamma\left(\frac{z - \alpha}{z - 1/\bar{\alpha}}\right) = \gamma\bar{\alpha}\left(\frac{z - \alpha}{\bar{\alpha} z - 1}\right)$$

NOW RECALL FROM ABOVE THAT IF $|z| = 1 \longrightarrow \left|\frac{z - \alpha}{\bar{\alpha} z - 1}\right| = 1$.

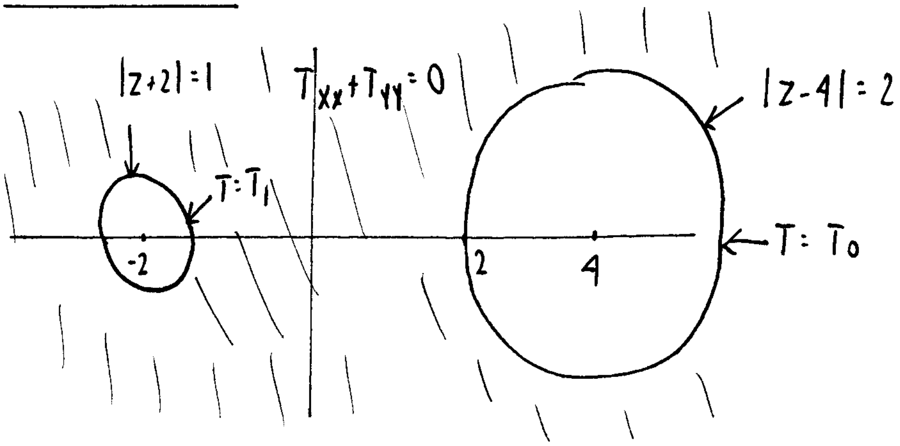
HENCE FOR $|w| = 1 \longrightarrow |\gamma\bar{\alpha}| = 1$ OR IF WE LET

$\beta = \gamma\bar{\alpha}$ WE REQUIRE $|\beta| = 1$

$$\Longrightarrow \quad w = \beta\left(\frac{z - \alpha}{\bar{\alpha} z - 1}\right) \qquad |\alpha| < 1, \quad |\beta| = 1.$$

∎

EXAMPLE 5 SOLVE LAPLACE'S EQUATION IN REGION SHOWN



$$\alpha^* = a + \frac{R^2}{(\bar{\alpha} - \bar{a})}$$

STEP 1 CHOOSE z_1 INSIDE $|z+2| \le 1$ AND z_2 INSIDE $|z-4| \le 2$.

CLEARLY WE WILL FIND z_1, z_2 REAL

FOR z_1, z_2 SYMMETRIC WRT $|z+2| \le 1 \longrightarrow$ $z_2 = -2 + \dfrac{1}{z_1 + 2}$

FOR z_1, z_2 SYMMETRIC WRT $|z-4| \le 2 \longrightarrow$ $z_1 = 4 + \dfrac{4}{z_2 - 4}$

SO $(z_2 + 2)(z_1 + 2) = 1$

$(z_1 - 4)(z_2 - 4) = 4 \longrightarrow ((z_1 + 2) - 6) = \dfrac{4}{z_2 - 4}$

SO $z_1 + 2 = 6 + \dfrac{4}{z_2 - 4}$

HENCE $(z_2 + 2)\left(6 + \dfrac{4}{(z_2 - 4)}\right) = 1$

SO $(z_2 + 2)(6(z_2 - 4) + 4) = (z_2 - 4)$

$\rightarrow 6(z_2^2 - 2z_2 - 8) + 4z_2 + 8 = z_2 - 4 \rightarrow 6z_2^2 - 9z_2 - 36 = 0.$

SO $2z_2^2 - 3z_2 - 12 = 0$ $\qquad z_2 = \dfrac{3 \pm \sqrt{9 + 4(12)(2)}}{4}$

$z_2 = \dfrac{3 \pm \sqrt{105}}{4}$ $\qquad \left\{ \begin{array}{l} z_2 = \dfrac{3 + \sqrt{105}}{4} \quad \text{(NEED + ROOT)} \\ z_1 = 4 + \dfrac{4}{z_2 - 4} \end{array} \right.$

MAP $z = z_1 \longrightarrow w = 0$

$z = z_2 \longrightarrow w = \infty$.