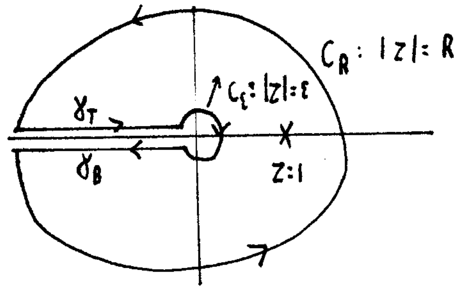
INTEGRALS OF MULTI-VALUED FUNCTIONS

PROBLEM 1 $\qquad I = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx \qquad 0<\alpha<1.$

TAKE THE INTEGRAL $\int_C \frac{z^{\alpha-1}}{1-z}\,dz$ WHERE C IS THE CONTOUR SHOWN BELOW. TAKE PRINCIPAL BRANCH OF $z^{\alpha-1}$. THEN,



$$\left|\int_{C_R}\frac{z^{\alpha-1}}{1-z}\,dz\right| \le a_0 R\,\frac{R^{\alpha-1}}{R} = a_0 R^{\alpha-1} \to 0 \text{ as } R\to\infty$$

SINCE $\alpha<1$

$$\left|\int_{C_\varepsilon}\frac{z^{\alpha-1}}{1-z}\,dz\right| \le a_0\,\varepsilon\,\frac{\varepsilon^{\alpha-1}}{1} = a_0\varepsilon^{\alpha} \to 0 \text{ as } \varepsilon\to 0$$

SINCE $\alpha>0$.

THUS $\displaystyle \lim_{R\to\infty}\int_{C_R}\frac{z^{\alpha-1}}{1-z}\,dz + \lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = 2\pi i\,\mathrm{RES}\left[\frac{z^{\alpha-1}}{1-z};1\right]$

SO (*) $\displaystyle \int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz + \int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = -2\pi i\,(1)^{\alpha-1} = -2\pi i.$

ON γ_T: $\quad z = \rho e^{i\pi} \qquad 0<\rho<\infty.$

SO $\quad 1-z = 1+\rho \quad$ (this is why we took $1-z$ rather than $1+z$)

$\quad dz = e^{i\pi}\,d\rho.$

SO $\displaystyle \int_{\gamma_T}\frac{z^{\alpha-1}}{1-z}\,dz = \int_\infty^0 \frac{e^{i\pi}\rho^{\alpha-1}}{1+\rho}\,e^{i\pi(\alpha-1)}\,d\rho = e^{i\pi(\alpha-1)}\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\,d\rho$

ON γ_B: $\quad z = \rho e^{-i\pi} \qquad 0<\rho<\infty \qquad 1-z = 1+\rho \qquad dz = e^{-i\pi}\,d\rho$

$$\int_{\gamma_B}\frac{z^{\alpha-1}}{1-z}\,dz = \int_0^\infty \frac{e^{-i\pi}\rho^{\alpha-1}e^{-i\pi(\alpha-1)}}{1+\rho}\,d\rho = -e^{-i\pi(\alpha-1)}\int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho$$

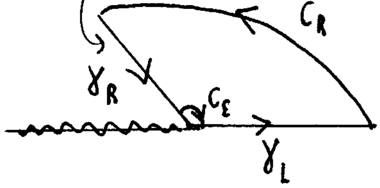
substitute these integrals into (*). Then,

$$\left[e^{i\pi(\alpha-1)} - e^{-i\pi(\alpha-1)}\right]\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = -2\pi i. \qquad \int_0^\infty\frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = \frac{-2\pi i}{\left(e^{i\pi(\alpha-1)} - e^{-i\pi(\alpha-1)}\right)}$$

THUS $\displaystyle \int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\,d\rho = \frac{-2\pi i}{2i\sin[\pi(\alpha-1)]} = \frac{\pi}{\sin[\pi(1-\alpha)]}$

PROBLEM 2 $\quad I = \int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\,dx \qquad 0<\alpha<3.$ $\qquad z = \rho e^{2\pi i/3}.$

TAKE THE CONTOUR $\displaystyle\int_C \frac{z^{\alpha-1}}{1+z^3}\,dz$



TAKE BRANCH CUT ON negative Real axis.

THEN $\displaystyle\left|\int_{C_\varepsilon} \frac{z^{\alpha-1}}{1+z^3}\,dz\right| \le a_0 \underbrace{\varepsilon}_{\text{length of curve}} \varepsilon^{\alpha-1} = a_0\varepsilon^{\alpha} \to 0$ as $\varepsilon\to 0$ since $\alpha>0$.

$$\left|\int_{C_R} \frac{z^{\alpha-1}}{1+z^3}\,dz\right| \le a_0 R\,\frac{R^{\alpha-1}}{R^3} = a_0 R^{\alpha-3} \to 0 \text{ As } R\to\infty \text{ since } \alpha<3.$$

NOW inside C: $1+z^3=0$ WHEN $z = e^{i\pi/3}$.

THUS $\displaystyle \operatorname{RES}\left[\frac{z^{\alpha-1}}{1+z^3};\, e^{i\pi/3}\right] = \frac{e^{i\pi/3(\alpha-1)}}{3e^{2\pi i/3}} = \frac{1}{3}e^{\alpha\pi i/3}e^{-i\pi} = -\frac{1}{3}e^{\alpha\pi i/3}.$

NEXT $\displaystyle \lim_{R\to\infty}\int_{C_R} + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} + \int_{\gamma_R} + \int_{\gamma_L} = 2\pi i \operatorname{RES}\left(\frac{z^{\alpha-1}}{1+z^3};\, e^{\pi i/3}\right) = -\frac{2\pi i}{3}e^{\alpha\pi i/3}$

letting $R\to\infty$, $\varepsilon\to 0$ $\quad (*) \quad \displaystyle\int_{\gamma_R} + \int_{\gamma_L} = -\frac{2\pi i}{3}e^{\alpha\pi i/3}.$

NOW ON γ_R: $\quad z = \rho e^{2\pi i/3} \qquad dz = e^{2\pi i/3}\,d\rho \qquad 1+z^3 = 1+\rho^3.$

so $\displaystyle\int_{\gamma_R} = \int_\infty^0 \frac{\rho^{\alpha-1}}{1+\rho^3} e^{2\pi i(\alpha-1)/3} e^{2\pi i/3}\,d\rho = -e^{2\pi i\alpha/3}\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho.$

NOW ON γ_L $\qquad z=\rho \qquad dz = d\rho \qquad \displaystyle\int_{\gamma_L}\frac{z^{\alpha-1}}{1+z^3}\,dz = \int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho.$

substituting these integrals INTO $(*)$ we get

$$\left(1-e^{+2\pi i\alpha/3}\right)\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^2}\,d\rho = -\frac{2\pi i}{3}e^{\alpha\pi i/3}.$$

so $\displaystyle\left(e^{-\alpha\pi i/3} - e^{\alpha\pi i/3}\right) I = -\frac{2\pi i}{3} \longrightarrow +2i\sin\left(-\alpha\pi/3\right) I = -2\pi i/3$

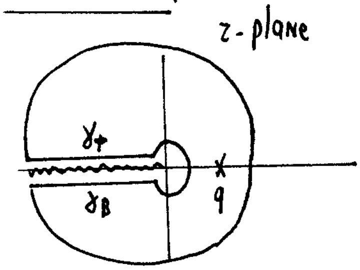
OR $\displaystyle\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho^3}\,d\rho = \frac{\pi}{3\sin(\pi\alpha/3)}$

PROBLEM 3 WHAT CONTOURS SHOULD WE TAKE FOR CALCULATING

$$I_1 = \int_0^\infty \frac{x^\alpha}{(x+q)^2}\,dx \qquad I_2 = \int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dz \qquad I_3 = P.V.\int_0^\infty \frac{x^\alpha}{x^2-1}\,dx$$

$-1 < \alpha < 1$ $-1 < \alpha < 3$ $-1 < \alpha < 1$

INTEGRAL I_1



$$\int_C \frac{z^\alpha}{(q-z)^2}\,dz \qquad \text{TAKE } z^\alpha \text{ with branch cut shown.}$$

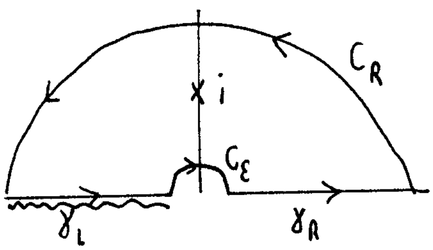
ON γ_T : $z = \rho e^{i\pi}$ so $(q-z)^2 = (q+\rho)^2$

ON γ_B : $z = \rho e^{-i\pi}$ so $(q-z)^2 = (q+\rho)^2$

this contour will work.

INTEGRAL I_2 WE could take precisely the same contour as for I_1 above.
This would give poles at $z = \pm i$ when integrating $\int_C \frac{z^\alpha}{(z^2+1)^2}\,dz$.

HOWEVER, PERHAPS IT IS EASIER TO TAKE CONTOUR BELOW:



$\int_C \frac{z^\alpha}{(z^2+1)^2}\,dz$ with taking principal branch FOR z^α.

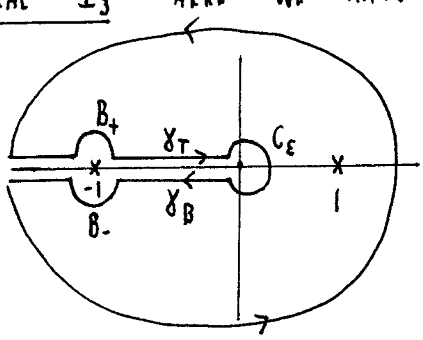
ON γ_L : $z = \rho e^{i\pi}$ so $1+z^2 = 1+\rho^2$.

ON γ_R : $z = \rho$ so $1+z^2 = \rho^2$. $i^\alpha = e^{\frac{i\pi}{2}\alpha}$.

NOW $$\text{RES}\left[\frac{z^\alpha}{(z^2+1)^2}\,;\,i\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{(z+i)^2 z^\alpha}{(z+i)^2(z-i)^2}\right]$$

$$= \lim_{z\to i}\frac{d}{dz}\left[(z-i)^{-2}z^\alpha\right] = \alpha z^{\alpha-1}(z-i)^{-2}\Big|_i - 2(z-i)^{-3}z^\alpha\Big|_i$$

INTEGRAL I_3 HERE WE HAVE NO CHOICE BUT TO TAKE CONTOUR



$\int_C \frac{z^\alpha}{z^2-1}\,dz$ pole at ± 1 along Real axis

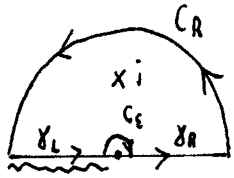
NOTE $\int_{B_\pm} \not\to 0$ AS $\varepsilon \to 0$.

ON B_+ : $z = -1 + \varepsilon e^{i\varphi}$ $\pi < \varphi < 0$

$$\lim_{\varepsilon\to 0}\int_{B_+}\frac{z^\alpha}{z^2-1}\,dz \to \int_\pi^0 \frac{e^{i\pi\alpha}}{\varepsilon e^{i\varphi}(-2)}\,\varepsilon i e^{i\varphi}\,d\varphi \to \frac{i\pi}{2}e^{i\pi\alpha}.$$

PROBLEM 4 $\qquad I = \displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,dx.$

CONSIDER THE CONTOUR



$$\int_C \frac{\operatorname{LOG} z}{z^2+1}\,dz.$$

HERE $\operatorname{LOG}(z)$ is principal branch.

$$\left|\int_{C_R} \frac{\operatorname{LOG} z}{z^2+1}\,dz\right| \leq a_0 R\,\frac{\ln R}{R^2} \to 0 \text{ as } R\to\infty$$

$$\left|\int_{C_\varepsilon} \frac{\operatorname{LOG} z}{z^2+1}\,dz\right| \leq a_0 \varepsilon\,\frac{\ln \varepsilon}{1} \to 0 \text{ AS } \varepsilon\to 0.$$

NOW $\displaystyle\lim_{R\to\infty}\int_{C_R} + \lim_{\varepsilon\to 0}\int_{C_\varepsilon} + \int_{\gamma_L} + \int_{\gamma_R} = 2\pi i \operatorname{RES}\left[\frac{\operatorname{LOG} z}{z^2+1};\ i\right]$

LET $R\to\infty$, $\varepsilon\to 0$ TO GET (*) $\displaystyle\int_{\gamma_L} + \int_{\gamma_R} = 2\pi i \operatorname{RES}\left[\frac{\operatorname{LOG} z}{z^2+1};\ i\right] = 2\pi i\,\frac{\operatorname{LOG} i}{2i} = \pi\,(i\pi$

$= i\pi^2/2.$

<u>NOW ON γ_L:</u> $\quad z = p e^{i\pi}. \quad dz = e^{i\pi}\,dp. \qquad z^2+1 = p^2+1.$

$\operatorname{LOG} z = \ln p + i\pi$

SO $\displaystyle\int_{\gamma_L} = \int_\infty^0 \left(\frac{\ln p + i\pi}{p^2+1}\right) e^{i\pi}\,dp = \int_0^\infty \frac{\ln p + i\pi}{p^2+1}\,dp.$

<u>NOW ON γ_R:</u> $\quad z = p \quad$ SO $\quad dz = dp \qquad z^2+1 = p^2+1$

$\operatorname{LOG} z = \ln p$

SO $\displaystyle\int_{\gamma_R} = \int_0^\infty \frac{\ln p}{p^2+1}\,dp.$

substitute $\int_{\gamma_L}$ AND $\int_{\gamma_R}$ INTO (*) TO get $\displaystyle\int_0^\infty \frac{\ln p}{p^2+1}\,dp + \int_0^\infty \frac{\ln p}{p^2+1}\,dp + i\pi\int_0^\infty \frac{1}{1+p^2}\,dp$

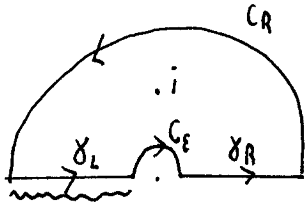
$= i\pi^2/2.$

EQUATING REAL AND IMAGINARY PARTS:

$$\int_0^\infty \frac{\ln p}{p^2+1}\,dp = 0 \quad \text{AND} \quad \int_0^\infty \frac{1}{1+p^2}\,dp = \pi/2.$$

PROBLEM 5 $\quad I = \int_0^\infty \frac{(\ln r)^2}{1+r^2}\, dr.$ $\qquad$ $\operatorname{LOG} Z$ is principal value.

TAKE $\displaystyle\int_C \frac{(\operatorname{LOG} Z)^2}{1+Z^2}\, dZ$ ON



NOW $\displaystyle \operatorname{RES}\left[\frac{(\operatorname{LOG} Z)^2}{1+Z^2};\ i\right] = \frac{(\operatorname{LOG} i)^2}{2i} = \frac{(i\pi/2)^2}{2i} = -\frac{\pi^2}{8i} = \frac{i\pi^2}{8}$

THUS AGAIN $\displaystyle\int_{C_R} \to 0$ AND $\displaystyle\int_{C_\epsilon} \to 0$. SO

$$(*) \quad \int_{\gamma_L} + \int_{\gamma_R} = 2\pi i \operatorname{RES}(\ ;\ i) = 2\pi i\left(i\pi^2/8\right) = -\pi^3/4.$$

NOW ON γ_L: $\quad Z = pe^{i\pi} \quad 0<p<\infty \quad 1+Z^2 = 1+p^2 \quad dZ = e^{i\pi}\, dp$

$$(\log Z)^2 = (\ln p + i\pi)^2 = (\ln p)^2 + 2i\pi \ln p - \pi^2.$$

$$\text{THUS} \quad \int_{\gamma_L} = \int_\infty^0 \left(\frac{(\ln p)^2 + 2i\pi \ln p - \pi^2}{1+p^2}\right) e^{i\pi}\, dp = \int_0^\infty \frac{(\ln p)^2 + 2i\pi \ln p - \pi^2}{1+p^2}\, dp$$

NOW ON γ_R: $\quad Z = p$

$$\int_{\gamma_R} = \int_0^\infty \frac{(\ln p)^2}{1+p^2}.$$

substitute $\displaystyle\int_{\gamma_L}, \int_{\gamma_R}$ INTO $(*)$ to get:

$$2\int_0^\infty \frac{(\ln p)^2}{1+p^2}\, dp + 2i\pi \int_0^\infty \frac{\ln p}{1+p^2}\, dp - \pi^2 \int_0^\infty \frac{1}{1+p^2}\, dp = -\pi^3/4.$$

EQUATING REAL AND IMAGINARY PARTS:

$$\int_0^\infty \frac{\ln p}{1+p^2}\, dp = 0 \quad \text{AND} \quad \int_0^\infty \frac{(\ln p)^2}{1+p^2}\, dp = \frac{\pi^2}{2}\int_0^\infty \frac{1}{1+p^2}\, dp - \frac{\pi^3}{8}$$

$$= \frac{\pi^2}{2} \operatorname{TAN}^{-1} p \Big|_0^\infty - \frac{\pi^3}{8} = \frac{\pi^2}{2}\left(\frac{\pi}{2}\right) - \frac{\pi^3}{8}$$

$$\text{so} \quad \int_0^\infty \frac{(\ln p)^2}{1+p^2}\, dp = \pi^3/8.$$