Homework 3a

Math 615

March 13, 2024

The geometry of $I_n(X, d[\mathbb{P}^1])$ for small n and d.

Let $X = \mathrm{Total}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$ be the resolved conifold and let $\mathbb{P}^1 \subset X$ be the zero section.

1. Show that $I_2(X, [\mathbb{P}^1]) \cong \mathrm{Bl}_{\mathbb{P}^1}(X)$, the blowup of X along $\mathbb{P}^1$.
2. Show that $I_3(X, 2[\mathbb{P}^1]) \cong \mathbb{P}^1$.
3. Show that $I_4(X, 2[\mathbb{P}^1])$ has a component birational to $\mathbb{P}^3$ and a component birational to $X \times \mathbb{P}^1$.

An adequate solution to the above problems will consist of constructing a natural bijection between the points of the moduli space and the points of the given space. A better solution will be to construct an actual morphism between the spaces by constructing a flat family of subschemes over the given space (thus inducing a map to the moduli space).

Hints:

1. Remember that X has a dominant map to an affine variety – it is a resolution of singularities of the conifold singularity $\{xy = zw\} \subset \mathbb{C}^4$.
2. For problems 2 and 3, you will want to show that a pure subscheme (no embedded points) in the class $2[\mathbb{P}^1]$ is contained in a smooth surface $L \subset X$ where L is the total space of a subline bundle $L \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.
3. A pure dimension 1 subscheme of a smooth surface is given by a principle ideal (so it is determined by its divisor) and the holomorphic Euler characteristic can be determined by the adjunction formula on that surface.