Midterm 2 — June 14th, 2017 Duration: 50 minutes

This test has 4 ***questions*** *on* ***9 pages****, for a total of 30 points.*

- Read all the questions carefully before starting to work.
- All questions are long-answer; you should give complete arguments and explanations for all your calculations. Write legibly and in a coherent order.
- Continue on the back of the previous page if you run out of space or use the blank page at the end. If you continue a problem on a different page, indicate this clearly at the bottom of the problem's original page.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

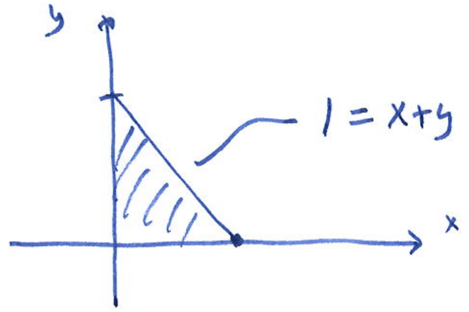
First Name: Solutions Last Name: ______________

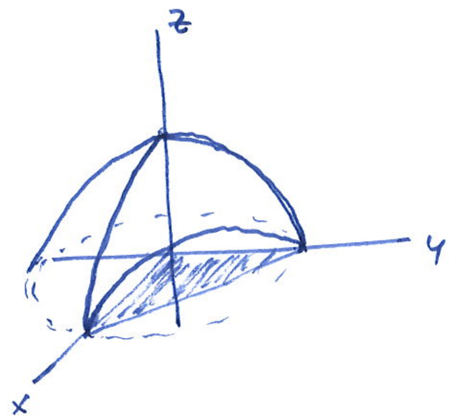
Student-No: ______________ Class (253 or 200): ______________

Signature: ______________

Question:	1	2	3	4	Total
Points:	4	5	11	10	30
Score:					

4 marks 1. Find the volume of the region lying below the paraboloid $z = 1 - x^2 - y^2$ and above the triangle in the xy plane having vertices $(0,0)$, $(1,0)$, and $(0,1)$.





$$\text{Volume} = \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x^2-y^2)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[y(1-x^2) - \frac{1}{3}y^3 \right]_0^{1-x} dx$$

$$= \int_{x=0}^{1} (1-x)(1-x^2) - \frac{1}{3}(1-x)^3 \, dx$$

$$= \int_{x=0}^{1} 1 - x - x^2 + x^3 - \frac{1}{3}\left[1 - 3x + 3x^2 - x^3\right] dx$$

$$= \int_{x=0}^{1} \left[\frac{2}{3} - 2x^2 + \frac{4}{3}x^3\right] dx$$

$$= \left[\frac{2}{3}x - \frac{2}{3}x^3 + \frac{1}{3}x^4\right]_0^1 = \frac{2}{3} - \frac{2}{3} + \frac{1}{3}$$

$$\boxed{= \frac{1}{3}}$$

5 marks 2. Let C be the curve given by the intersection of the paraboloid $z = x^2 + y^2$ and the plane $2x + 4y + 5z = 3$. Find the coordinates of the point on C which is the closest to the origin. [Suggested method: minimize the distance squared instead of distance, eliminate z from the equations, then use Lagrange multipliers.]

Minimize the function $f(x,y,z) = x^2+y^2+z^2$

subject to $z = x^2+y^2$ and $2x+4y+5z-3=0$

$\Leftrightarrow$ minimize $f(x,y) = x^2+y^2+(x^2+y^2)^2$

subject to $g(x,y) = 2x+4y+5(x^2+y^2)-3=0$

① $f_x = 2x + 4x(x^2+y^2) = \lambda(2+10x) = \lambda g_x$

② $f_y = 2y + 4y(x^2+y^2) = \lambda(4+10y) = \lambda g_y$

multiply ① by y and ② by x and subtract:

$$\begin{aligned} 2xy + 4xy(x^2+y^2) &= \lambda(2y+10xy) \\ -2xy - 4xy(x^2+y^2) &= \lambda(-4x-10xy) \\ \hline 0 &= \lambda(2y-4x) \end{aligned}$$

$\left\{ \lambda \neq 0 \text{ since if it were } ① \,\&\, ② \Rightarrow x=y=0 \text{ and point doesn't lie on } C. \right\}$

so $0 = 2y-4x \Rightarrow y = 2x$ sub into $g=0$:

$2x + 8x + 5(x^2+4x^2) - 3 = 0$

$25x^2 + 10x - 3 = 0 \Rightarrow (5x+3)(5x-1) = 0$

$\Rightarrow x = \frac{1}{5}$ or $-\frac{3}{5}$ $\quad x=\frac{1}{5} \Rightarrow y = \frac{2}{5} \Rightarrow z = \frac{1}{25}(1+2^2) = \frac{1}{5}$

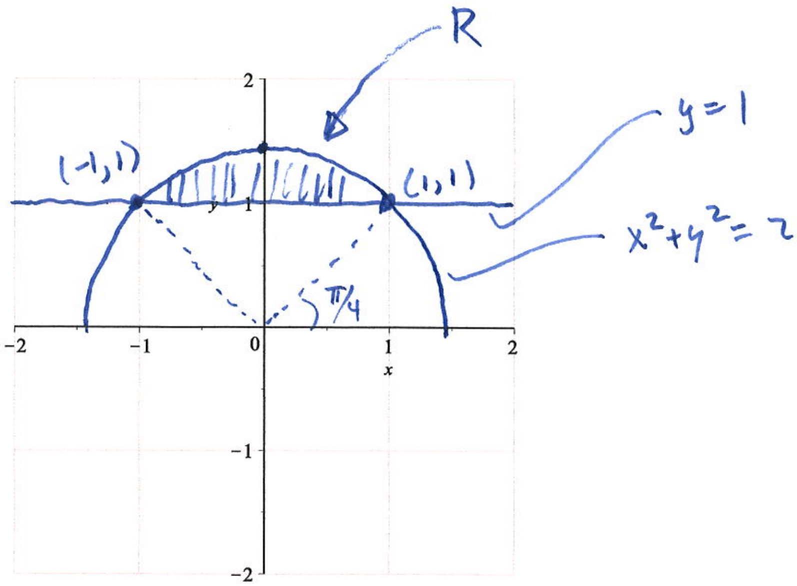
$x = -\frac{3}{5} \Rightarrow y = -\frac{6}{5} \Rightarrow z = \frac{1}{25}(3^2+6^2) = \frac{9}{25}(1+4) = \frac{9}{5}$ so our two points are $\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$ and $\left(-\frac{3}{5}, -\frac{6}{5}, \frac{9}{5}\right)$

the closer one is $\boxed{\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)}$

3. Let R be the region which is contained inside the circle of radius $\sqrt{2}$ centered at the origin, but is above the line $y = 1$. Let

$$f(x, y) = \frac{y}{x^2 + y^2}.$$

2 marks (a) sketch the region R on the axis provided.



2 marks (b) Set up the double integral $\iint_R f dA$ as an iterated integral with the x integral as the inner integral. **Do not evaluate** (yet!) the integral.

$$\int_{y=1}^{\sqrt{2}} \int_{x=-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \frac{y}{x^2+y^2} \, dx \, dy$$

2 marks (c) Set up the double integral $\iint_R f dA$ as an iterated integral with the y integral as the inner integral. **Do not evaluate** (yet!) the integral.

$$\int_{x=-1}^{1} \int_{y=1}^{\sqrt{2-x^2}} \frac{y}{x^2+y^2}\, dy\, dx$$

2 marks (d) Set up the double integral $\iint_R f dA$ in polar coordinates. **Do not evaluate** (yet!) the integral.

the line $y=1$ is $r\sin\theta = 1$ or $r = \frac{1}{\sin\theta}$ in polar coords

$$\int_{\theta=\pi/4}^{3\pi/4} \int_{r=\frac{1}{\sin\theta}}^{\sqrt{2}} \frac{r\sin\theta}{r^2} \cdot r\, dr\, d\theta$$

$$= \int_{\theta=\pi/4}^{3\pi/4} \int_{r=\frac{1}{\sin\theta}}^{\sqrt{2}} \sin\theta\, dr\, d\theta$$

3 marks (e) Evaluate the double integral $\iint_R f dA$ by any method.

via polar coords: (recommended) $\displaystyle\int_{\theta=\pi/4}^{3\pi/4}\int_{r=\frac{1}{\sin\theta}}^{\sqrt{2}} \sin\theta \, dr\, d\theta$

$$= \int_{\theta=\pi/4}^{3\pi/4} \sin\theta\left[\sqrt{2}-\frac{1}{\sin\theta}\right]d\theta = \int_{\pi/4}^{3\pi/4}(\sqrt{2}\sin\theta - 1)\,d\theta$$

$$= -\sqrt{2}\cos\theta - \theta\Big]_{\pi/4}^{3\pi/4} = -\sqrt{2}\left[-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right] - \left(\frac{3\pi}{4}-\frac{\pi}{4}\right) = \boxed{2-\frac{\pi}{2}}$$

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via dydx (not recommended)

$$\int_{x=-1}^{1}\int_{y=1}^{\sqrt{2-x^2}} \frac{y}{x^2+y^2}\,dy\,dx = \int_{x=-1}^{1} \frac{1}{2}\log(x^2+y^2)\Big|_{y=1}^{\sqrt{2-x^2}} dx = \int_{x=-1}^{1}\left(\frac{1}{2}\log 2 - \frac{1}{2}\log(1+x^2)\right)dx$$

$$= \log 2 - \frac{1}{2}\int_{x=-1}^{1}\log(1+x^2)\,dx$$

$u = \log(1+x^2)$ $dv = dx$

$du = \frac{2x}{1+x^2}dx$ $v = x$

$$= \log 2 - \frac{1}{2}\left[x\log(1+x^2)\Big|_{-1}^{1} - \int_{x=-1}^{1}\frac{2x^2}{1+x^2}dx\right]$$

$$= \log 2 - \frac{1}{2}(2\log 2) + \int_{x=-1}^{1}\frac{x^2}{1+x^2}dx$$

$$\left[\begin{array}{l} x = \tan u \quad dx = \sec^2 u\, du \\ 1+x^2 = 1+\tan^2 u = \sec^2 u \\ x = \pm 1 \Rightarrow u = \pm\frac{\pi}{4}\end{array}\right]$$

$$= \int_{x=-1}^{1}\frac{x^2\,dx}{1+x^2} = \int_{u=-\pi/4}^{\pi/4}\frac{\tan^2 u \sec^2 u\, du}{\sec^2 u} = \int_{u=-\pi/4}^{\pi/4}\tan^2 u\, du$$

$$= \int_{u=-\pi/4}^{\pi/4}(\sec^2 u - 1)\,du = \tan u - u\Big]_{-\pi/4}^{\pi/4} = \left(1-\frac{\pi}{4}\right) - \left(-1+\frac{\pi}{4}\right)$$

$$= 2 - \frac{\pi}{2}$$

4. Let $f(x, y) = (x^2 + y^2 - 2)(x - y)$

4 marks (a) Find all critical points of $f(x, y)$ and classify them as "local maximum", "local minimum", "saddle point", or "not an ordinary critical point".

$$\left.\begin{aligned} f_x &= 2x(x-y) + (x^2+y^2-2) = 0 \\ f_y &= 2y(x-y) - (x^2+y^2-2) = 0 \end{aligned}\right\} \text{solve}$$

adding them together: $2(x+y)(x-y) = 0 \Rightarrow x = y$ or $x = -y$

if $x = y$ then $y^2 + y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$(1,1)$ & $(-1,-1)$ are crit pts

if $x = -y$ then $2y(-y-y) - ((-y)^2 + y^2 - 2) = 0$

$-4y^2 - 2y^2 + 2 = 0 \Rightarrow y^2 = \frac{1}{3}$ $y = \pm\frac{1}{\sqrt{3}}$

$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are crit. pts

$f_x = 3x^2 + y^2 - 2 - 2xy$ $f_{xx} = 6x - 2y$

$f_y = -3y^2 - x^2 + 2 + 2xy$ $f_{yy} = -6y + 2x$

$f_{xy} = -2x + 2y$ $D = f_{xx}f_{yy} - f_{xy}^2 = 4(3x-y)(-3y+x) - 4(-x+y)^2$

$D(1,1) = 4 \cdot 2 \cdot (-2) - 4 \cdot 0 = -16$

$D(-1,-1) = 4 \cdot (-2)(2) - 4 \cdot 0 = -16$

$$\boxed{\left.\begin{matrix}(1,1)\\(-1,-1)\end{matrix}\right\} \text{saddle points}}$$

$D\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = 4\left(\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - 4\left(-\frac{2}{\sqrt{3}}\right)^2$

$= \frac{4 \cdot 16}{3} - \frac{16}{3} = 16$ $f_{xx}\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} + \frac{2}{\sqrt{3}} > 0$

$D\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 4\left(\frac{-3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\left(\frac{-3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) - 4\left(\frac{2}{\sqrt{3}}\right)^2 = 16$

$f_{xx}\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}} - \frac{2}{\sqrt{3}} < 0$

$$\boxed{\begin{matrix}\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ loc min}\\[4pt] \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ loc max}\end{matrix}}$$

3 marks

(b) On the axis below, sketch the contour plot of $f(x, y)$. Start by ploting the contour $f(x, y) = 0$ and then using your classification of critical points to fill in the rest.

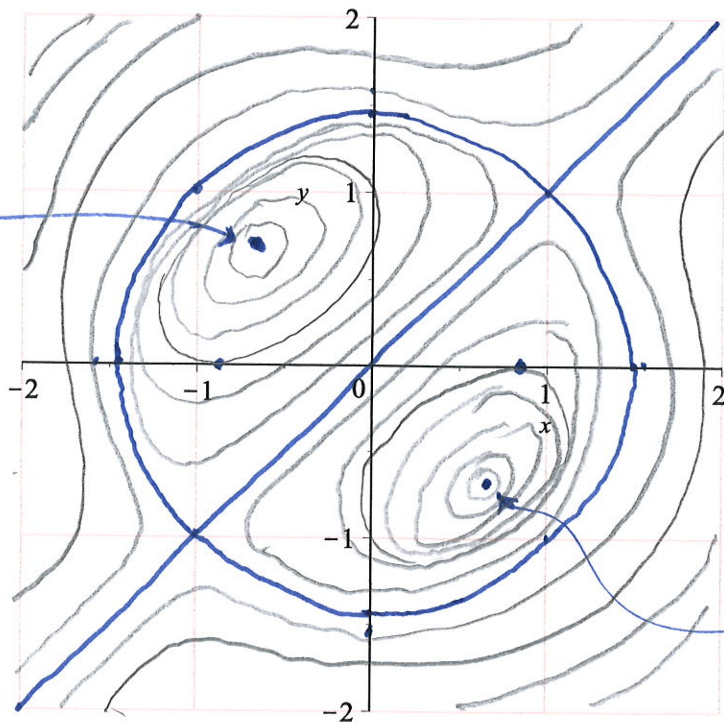
$f = 0$

$\Rightarrow x = y$

or $x^2+y^2=2$

local max

$\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$



$\left(\frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$

local min

3 marks

(c) Find the absolute maximum and minimum of $f(x, y)$ over the domain D where D is the region inside the circle of radius $\sqrt{2}$ centered at the origin and above the x-axis.

min/max must occur at a critical pt ~~inside~~ on the interior of D or on the boundary. $f=0$ on the circle of radius 2, $f\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) =$

$\left(\frac{1}{3}+\frac{1}{3}-2\right)\left(\frac{-2}{\sqrt{3}}\right) = \left(\frac{2}{3}-\frac{6}{3}\right)\left(\frac{-2}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}}$

on the part of the boundary of D given by the segment

$y=0 \quad -\sqrt{2} \leq x \leq \sqrt{2}$ f is given by $(x^2-2)x = g(x) = x^3-2x$

$g'(x) = 3x^2-2=0 \Rightarrow x^2 = \frac{2}{3} \quad x = \pm\frac{\sqrt{2}}{\sqrt{3}}$ so $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$ are

possible extrema. $f\left(\pm\sqrt{\frac{2}{3}}, 0\right) = \pm\sqrt{\frac{2}{3}}\left(\frac{2}{3}-2\right) = \pm\sqrt{\frac{2}{3}}\left(\frac{-4}{3}\right)$

$+\frac{4}{3}\sqrt{\frac{2}{3}}$ is not as big as $\frac{8}{3\sqrt{3}}$

$\left(\sqrt{\frac{2}{3}}, 0\right)$ absolute min, $\left(\frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ abs max

$f_{min} = \frac{-4}{3}\frac{\sqrt{2}}{\sqrt{3}} \quad f_{max} = \frac{8}{3\sqrt{3}}$

Extra Space.