

Midterm 2 — June 14th, 2017 Duration: 50 minutes*This test has 4 questions on 9 pages, for a total of 30 points.*

- Read all the questions carefully before starting to work.
- All questions are long-answer; you should give complete arguments and explanations for all your calculations. Write legibly and in a coherent order.
- Continue on the back of the previous page if you run out of space or use the blank page at the end. If you continue a problem on a different page, indicate this clearly at the bottom of the problem's original page.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: Solutions Last Name: _____

Student-No: _____ Class (253 or 200): _____

Signature: _____

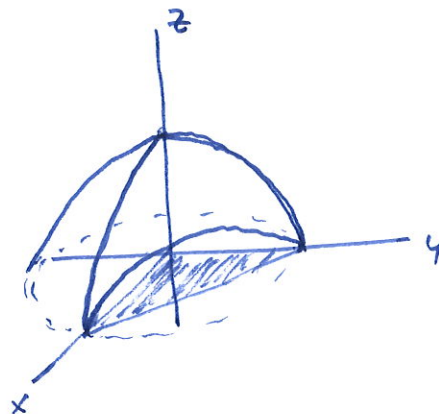
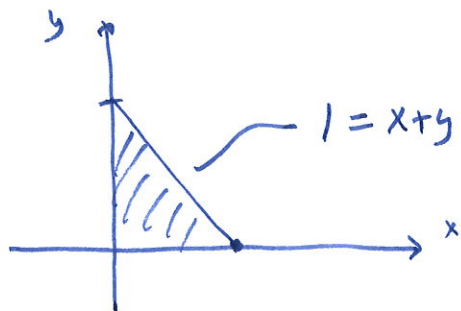
Question:	1	2	3	4	Total
Points:	4	5	11	10	30
Score:					

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - speaking or communicating with other examination candidates, unless otherwise authorized;
 - purposely exposing written papers to the view of other examination candidates or imaging devices;
 - purposely viewing the written papers of other examination candidates;
 - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

4 marks

1. Find the volume of the region lying below the paraboloid $z = 1 - x^2 - y^2$ and above the triangle in the xy plane having vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.



$$\text{Volume} = \int_{x=0}^1 \int_{y=0}^{1-x} (1 - x^2 - y^2) dy dx$$

$$= \int_{x=0}^1 \left[y(1 - x^2) - \frac{1}{3}y^3 \right]_0^{1-x} dx$$

$$= \int_{x=0}^1 (1-x)(1-x^2) - \frac{1}{3}(1-x)^3 dx$$

$$= \int_{x=0}^1 1 - x - x^2 + x^3 - \frac{1}{3}[1 - 3x + 3x^2 - x^3] dx$$

$$= \int_{x=0}^1 \left[\frac{2}{3} - 2x^2 + \frac{4}{3}x^3 \right] dx$$

$$= \left[\frac{2}{3}x - \frac{2}{3}x^3 + \frac{1}{3}x^4 \right]_0^1 = \frac{2}{3} - \frac{2}{3} + \frac{1}{3}$$

$$\boxed{= \frac{1}{3}}$$

5 marks

2. Let C be the curve given by the intersection of the paraboloid $z = x^2 + y^2$ and the plane $2x + 4y + 5z = 3$. Find the coordinates of the point on C which is the closest to the origin.
[Suggested method: minimize the distance squared instead of distance, eliminate z from the equations, then use Lagrange multipliers.]

Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$

subject to $z = x^2 + y^2$ and $2x + 4y + 5z - 3 = 0$

\Leftrightarrow minimize $f(x, y) = x^2 + y^2 + (x^2 + y^2)^2$

subject to $g(x, y) = 2x + 4y + 5(x^2 + y^2) - 3 = 0$

$$\textcircled{1} \quad f_x = 2x + 4x(x^2 + y^2) = \lambda(2 + 10x) = \lambda g_x$$

$$\textcircled{2} \quad f_y = 2y + 4y(x^2 + y^2) = \lambda(4 + 10y) = \lambda g_y$$

multiply $\textcircled{1}$ by y and $\textcircled{2}$ by x and subtract:

$$\begin{aligned} 2xy + 4xy(x^2 + y^2) &= \lambda(2y + 10xy) \\ -2xy - 4xy(x^2 + y^2) &= \lambda(-4x - 10xy) \end{aligned}$$

$$0 = \lambda(2y - 4x)$$

so $0 = 2y - 4x \Rightarrow y = 2x$ sub into $g=0$:

$\left\{ \begin{array}{l} \lambda \neq 0 \text{ since if it} \\ \text{were } \textcircled{1} \text{ \& } \textcircled{2} \Rightarrow x=y=0 \\ \text{and point doesn't} \\ \text{lie on } C. \end{array} \right\}$

$$2x + 8x + 5(x^2 + 4x^2) - 3 = 0$$

$$25x^2 + 10x - 3 = 0 \Rightarrow (5x + 3)(5x - 1) = 0$$

$$\Rightarrow x = \frac{1}{5} \text{ or } -\frac{3}{5} \quad x = \frac{1}{5} \Rightarrow y = \frac{2}{5} \Rightarrow z = \frac{1}{25}(1 + 2^2) = \frac{1}{5}$$

$$x = -\frac{3}{5} \Rightarrow y = -\frac{6}{5} \Rightarrow z = \frac{1}{25}(3^2 + 6^2) = \frac{9}{25}(1 + 4) = \frac{9}{5} \quad \text{so our two}$$

points are $\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$ and $\left(-\frac{3}{5}, -\frac{6}{5}, \frac{9}{5}\right)$

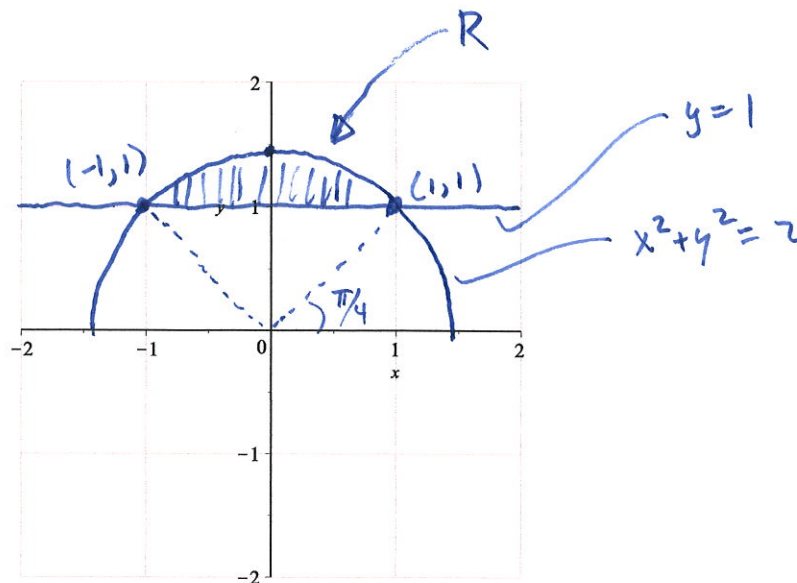
the closer one is $\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$

3. Let R be the region which is contained inside the circle of radius $\sqrt{2}$ centered at the origin, but is above the line $y = 1$. Let

$$f(x, y) = \frac{y}{x^2 + y^2}.$$

2 marks

- (a) sketch the region R on the axis provided.



2 marks

- (b) Set up the double integral $\iint_R f dA$ as an iterated integral with the x integral as the inner integral. **Do not evaluate** (yet!) the integral.

$$\int_{y=1}^{\sqrt{2}} \int_{x=-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \frac{y}{x^2+y^2} dx dy$$

2 marks

- (c) Set up the double integral $\iint_R f dA$ as an iterated integral with the y integral as the inner integral. **Do not evaluate** (yet!) the integral.

$$\int_{x=-1}^1 \int_{y=1}^{\sqrt{2-x^2}} \frac{y}{x^2+y^2} dy dx$$

2 marks

- (d) Set up the double integral $\iint_R f dA$ in polar coordinates. **Do not evaluate** (yet!) the integral.

The line $y=1$ is $r \sin \theta = 1$ or $r = \frac{1}{\sin \theta}$ in polar coords

$$\int_{\theta=\pi/4}^{3\pi/4} \int_{r=\frac{1}{\sin \theta}}^{\sqrt{2}} \frac{r \sin \theta}{r^2} r dr d\theta$$

$$= \int_{\theta=\pi/4}^{3\pi/4} \int_{r=\frac{1}{\sin \theta}}^{\sqrt{2}} \sin \theta dr d\theta$$

3 marks

(e) Evaluate the double integral $\iint_R f dA$ by any method.

via polar coords:
(recommended)

$$\int_{\theta=\pi/4}^{3\pi/4} \int_{r=\frac{1}{\sin\theta}}^{\sqrt{2}} \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=\pi/4}^{3\pi/4} \sin\theta \left[\sqrt{2} - \frac{1}{\sin\theta} \right] d\theta = \int_{\pi/4}^{3\pi/4} (\sqrt{2} \sin\theta - 1) d\theta$$

$$= -\sqrt{2} \cos\theta - \theta \Big|_{\pi/4}^{3\pi/4} = -\sqrt{2} \left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right] - \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) = \boxed{2 - \frac{\pi}{2}}$$

via $dy dx$ (not recommended)

$$\int_{x=-1}^1 \int_{y=1}^{\sqrt{2-x^2}} \frac{y}{x^2+y^2} dy dx = \int_{x=-1}^1 \left. \frac{1}{2} \log(x^2+y^2) \right|_{y=1}^{\sqrt{2-x^2}} dx = \int_{x=-1}^1 \left(\frac{1}{2} \log 2 - \frac{1}{2} \log(1+x^2) \right) dx$$

$$= \log 2 - \frac{1}{2} \int_{x=-1}^1 \log(1+x^2) dx$$

$u = \log(1+x^2) \quad dv = dx$
 $du = \frac{2x}{1+x^2} dx \quad v = x$

$$= \log 2 - \frac{1}{2} \left[x \log(1+x^2) \Big|_{-1}^1 - \int_{x=-1}^1 \frac{2x^2}{1+x^2} dx \right]$$

$$= \log 2 - \frac{1}{2} (2 \log 2) + \int_{x=-1}^1 \frac{x^2}{1+x^2} dx$$

$$= \int_{x=-1}^1 \frac{x^2 dx}{1+x^2} = \int_{u=-\pi/4}^{\pi/4} \frac{\tan^2 u \sec^2 u du}{\sec^2 u} = \int_{u=-\pi/4}^{\pi/4} \tan^2 u du$$

$$= \int_{u=-\pi/4}^{\pi/4} (\sec^2 u - 1) du = \tan u - u \Big|_{-\pi/4}^{\pi/4} = \left(1 - \frac{\pi}{4} \right) - \left(-1 + \frac{\pi}{4} \right)$$

$$= 2 - \frac{\pi}{2}$$

4. Let $f(x, y) = (x^2 + y^2 - 2)(x - y)$

4 marks

(a) Find all critical points of $f(x, y)$ and classify them as "local maximum", "local minimum", "saddle point", or "not an ordinary critical point".

$$\left. \begin{aligned} f_x &= 2x(x-y) + (x^2 + y^2 - 2) = 0 \\ f_y &= 2y(x-y) - (x^2 + y^2 - 2) = 0 \end{aligned} \right\} \text{ solve}$$

adding them together: $2(x+y)(x-y) = 0 \Rightarrow x=y \text{ or } x=-y$

if $x=y$ then $y^2 + y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$(1, 1)$ & $(-1, -1)$ are crit pts

if $x=-y$ then $2y(-y-y) - ((-y)^2 + y^2 - 2) = 0$

$$-4y^2 - 2y^2 + 2 = 0 \Rightarrow y^2 = \frac{1}{3} \quad y = \pm \frac{1}{\sqrt{3}}$$

$(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ are crit. pts

$$f_x = 3x^2 + y^2 - 2 - 2xy \quad f_{xx} = 6x - 2y$$

$$f_y = -3y^2 - x^2 + 2 + 2xy \quad f_{yy} = -6y + 2x$$

$$f_{xy} = -2x + 2y \quad D = f_{xx}f_{yy} - f_{xy}^2 = 4(3x-y)(-3y+x) - 4(-x+y)^2$$

$$D(1, 1) = 4 \cdot 2 \cdot (-2) - 4 \cdot 0 = -16$$

$$D(-1, -1) = 4 \cdot (-2) \cdot (2) - 4 \cdot 0 = -16$$

$$\left. \begin{aligned} (1, 1) \\ (-1, -1) \end{aligned} \right\} \text{ saddle points}$$

$$D\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = 4\left(\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - 4\left(-\frac{2}{\sqrt{3}}\right)^2$$

$$= \frac{4 \cdot 16}{3} - \frac{16}{3} = 16 \quad f_{xx}\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} + \frac{2}{\sqrt{3}} > 0$$

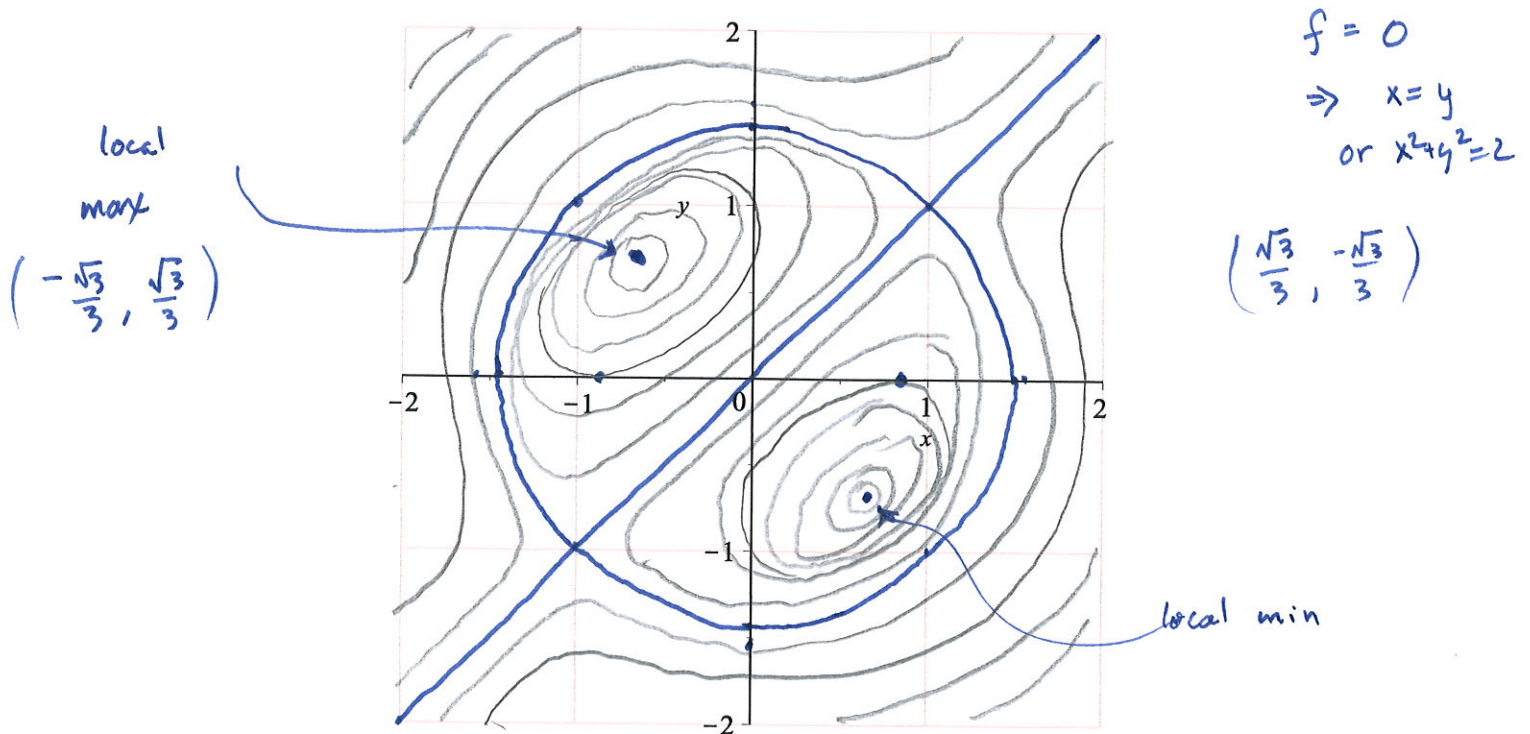
$$D\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 4\left(-\frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\left(-\frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) - 4\left(\frac{2}{\sqrt{3}}\right)^2 = 16$$

$$f_{xx}\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}} - \frac{2}{\sqrt{3}} < 0$$

$$\left. \begin{aligned} \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) & \text{ loc min} \\ \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) & \text{ loc max} \end{aligned} \right\}$$

3 marks

- (b) On the axis below, sketch the contour plot of $f(x, y)$. Start by plotting the contour $f(x, y) = 0$ and then using your classification of critical points to fill in the rest.



3 marks

- (c) Find the absolute maximum and minimum of $f(x, y)$ over the domain D where D is the region inside the circle of radius $\sqrt{2}$ centered at the origin and above the x -axis.

min/max must occur at a critical pt ~~inside~~ on the interior of D or on the boundary. $f=0$ on the circle of radius 2, $f(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = (\frac{1}{3} + \frac{1}{3} - 2)(\frac{-2}{\sqrt{3}}) = (\frac{2}{3} - \frac{6}{3})(\frac{-2}{\sqrt{3}}) = \frac{8}{3\sqrt{3}}$

on the part of the boundary of D given by the segment $y=0$ $-\sqrt{2} \leq x \leq \sqrt{2}$ f is given by $(x^2 - 2)x = g(x) = x^3 - 2x$

$g'(x) = 3x^2 - 2 = 0 \Rightarrow x^2 = \frac{2}{3} \quad x = \pm \frac{\sqrt{2}}{\sqrt{3}} \quad \text{so } (\pm \frac{\sqrt{2}}{\sqrt{3}}, 0) \text{ are possible extrema.}$

$f(\pm \frac{\sqrt{2}}{\sqrt{3}}, 0) = \pm \frac{\sqrt{2}}{\sqrt{3}} (\frac{2}{3} - 2) = \pm \frac{\sqrt{2}}{\sqrt{3}} (-\frac{4}{3})$

$+\frac{4}{3} \frac{\sqrt{2}}{\sqrt{3}}$ is not as big as $\frac{8}{3\sqrt{3}}$

$(\frac{\sqrt{2}}{\sqrt{3}}, 0)$ absolute min, $(\frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ abs max
 $f_{\min} = -\frac{4\sqrt{2}}{3\sqrt{3}} \quad f_{\max} = \frac{8}{3\sqrt{3}}$

Extra Space.