Final Exam
Math 200/253
June 27th, 2017

Last Name: Solutions First Name: ____________

Student # : ____________ Section (200 or 253) : ____________

Instructions:
No memory aids allowed. No calculators allowed. No communication devices allowed. Use the space provided on the exam. If you use the back of a page, write "see back" on the front of the page. This exam is 180 minutes long.

Question	Points	Score
1	6	
2	10	
3	14	
4	12	
5	6	
6	20	
7	6	
8	6	
Total:	80	

1. $\boxed{\text{6 points}}$ (a) Find the equation of the plane which contains the point $(2, 2, 0)$ and the line

$$x = 5t - 4$$
$$y = -4t + 5$$
$$z = t.$$

The vector $\langle 5, -4, 1\rangle$ is contained in the plane and since $(t=0)$ $(-4, 5, 0)$ is in the plane as is $(2,2,0)$ we have that $\langle 6, -3, 0\rangle$ is in the plane. Thus we get a normal to the plane:

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -3 & 0 \\ 5 & -4 & 1 \end{vmatrix} = \langle -3, -6, -9\rangle$$

$$-3(x-2) - 6(y-2) - 9z = 0$$

or $(x-2) + 2(y-2) + 3z = 0$

or $x + 2y + 3z = 6$

Answer:
$x+2y+3z=6$

(b) Find the x-intercept, the y-intercept, and the z-intercept of the plane (namely, the points where the plane hits the x-axis, the y-axis, the z-axis respectively).

get Intercepts by setting 2 of the three variables to 0

x intercept: $(6,0,0)$

y intercept: $(0,3,0)$

z interept: $(0,0,2)$

Answer:
$(6,0,0), (0,3,0)$ $(0,0,2)$

2. $f(x, y)$ is a function with

$$f(30, 50) = 40, \quad f_x(30, 50) = 2, \quad f_y(30, 50) = 3$$

(a) $\boxed{\text{3 points}}$ Find the equation of the plane tangent to the surface $z = f(x, y)$ at the point $(30, 50, 40)$.

$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Answer:

$z = 40 + 2(x-30) + 3(x-50)$

(b) $\boxed{\text{3 points}}$ Find the approximate value of $f(28, 53)$.

$$f(x,y) \approx 40 + 2(x-30) + 3(x-50)$$
$$= 40 + 2(28-30) + 3(53-50)$$
$$= 40 - 4 + 9 = 45$$

Answer: 45

(c) $\boxed{\text{4 points}}$ Let $g(s, t) = f(st, 5s + 4t)$. Compute $g_s(6, 5)$ and $g_t(6, 5)$.

$\frac{\partial x}{\partial s} = t \quad \frac{\partial y}{\partial s} = 5$

$\frac{\partial x}{\partial t} = s \quad \frac{\partial y}{\partial t} = 4$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2 \cdot \cancel{6}5 + 3 \cdot 5 = \cancel{24}\ 25$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 2 \cdot 6 + 3 \cdot 4 = 24$$

Answer:

$g_s = 25 \quad g_t = 24$

3. Let

$$f(x,y) = x^2 + y^2 - y - yx^2$$

and let

$$D = \{(x,y) : \quad x^2 + y^2 \leq 2\}$$

(a) $\boxed{\text{4 points}}$ Find all the critical points of $f(x,y)$.

$f_x = 2x - 2yx = 0 \Rightarrow x = 0$ or $2 - 2y = 0 \Rightarrow y = 1$

$f_y = 2y - 1 - x^2 = 0$ if $x = 0$ then $2y - 1 = 0 \Rightarrow y = \frac{1}{2}$

if $y = 1$ then $1 - x^2 = 0 \Rightarrow x = \pm 1$

Answer:

$(0, \frac{1}{2})$ $(1,1)$ $(-1,1)$

(b) $\boxed{\text{3 points}}$ Classify each of the critical points as a "local maximum", a "local minimum", a "saddle point", or "not an ordinary critical point".

$$D = f_{xx} f_{yy} - f_{xy}^2 = (2-2y)\cdot 2 - (-2x)^2$$

$$= 4(1 - y - x^2)$$

$D(0, \frac{1}{2}) = 4(1 - \frac{1}{2}) = 2 > 0$ $f_{yy} > 0$ $\Rightarrow$ $\boxed{(0, \frac{1}{2}) \text{ local min}}$

$D(1,1) = 4(1-1-1) = -4 < 0$ $\Rightarrow$ $\boxed{(1,1) \text{ saddle}}$

$D(-1,1) = 4(1-1-1) = -4 < 0$ $\Rightarrow$ $\boxed{(-1,1) \text{ saddle}}$

(c) $\boxed{\text{4 points}}$ Find the absolute maximum and absolute minimum of $f(x, y)$ on the domain D.

in addition to critical points on the interior, we check crit. pts on boundary $x^2+y^2=2$

$$\left(2x-2yx=\lambda(2x)\right)y \quad \Rightarrow \quad \cancel{2xy}-2y^2x=\cancel{2xy}-x-x^3$$

$$\left(2y-1-x^2=\lambda(2y)\right)x$$

$$\Downarrow$$

$$x^2+y^2=2 \qquad\qquad x^3+x-2y^2x=0$$

$$\Rightarrow x=0 \quad \text{or} \quad x^2+1-2y^2=0$$

if $x=0$ then $y=\pm\sqrt{2}$

$(0,\sqrt{2})$ $(0,-\sqrt{2})$

if $x^2+1-2y^2=0$, $x^2+y^2=2$ $\Rightarrow 1-3y^2=-2$

$3y^2=3$ $\quad y=\pm 1$

$\Rightarrow x=\pm 1$

boundary:

$f(0,\sqrt{2}) = 2-\sqrt{2}$

$f(0,-\sqrt{2}) = 2+\sqrt{2}$

$f(1,1) = 1+1-1-1=0$

$f(1,-1) = 1+1+1+1=4$

$f(-1,1) = 1+1-1-1=0$

$f(-1,-1) = 1+1+1+1=4$

maxes

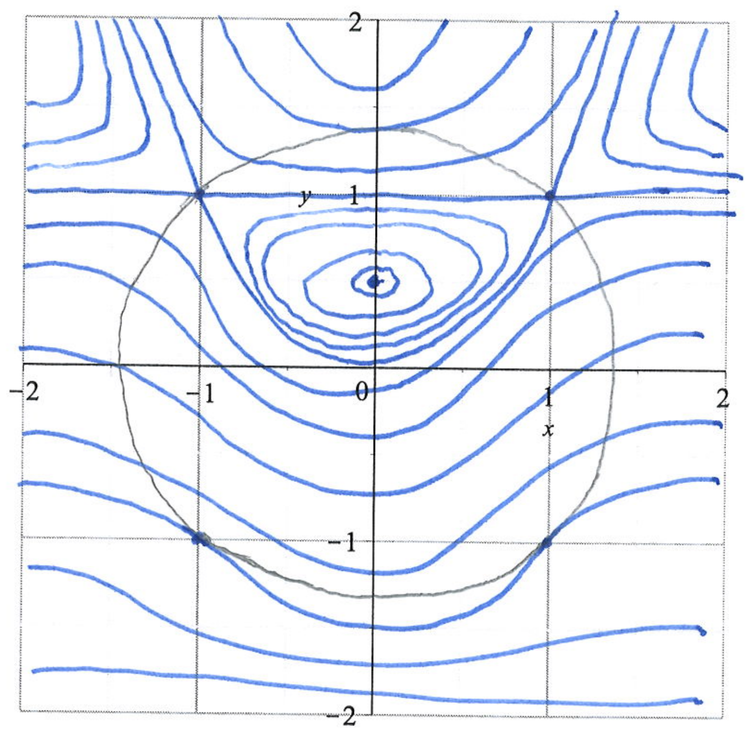
Interior

$f(0,\frac{1}{2}) = \frac{1}{4}-\frac{1}{2} = -\frac{1}{4}$ min

Answer: $(\pm1,-1)$ maxes

$(0,\frac{1}{2})$ min.

(d) $\boxed{\text{3 points}}$ Sketch the contour plot of f on the axes below. Hint: plot the contour $f(x, y) = 0$ by factoring f into a product of two terms.



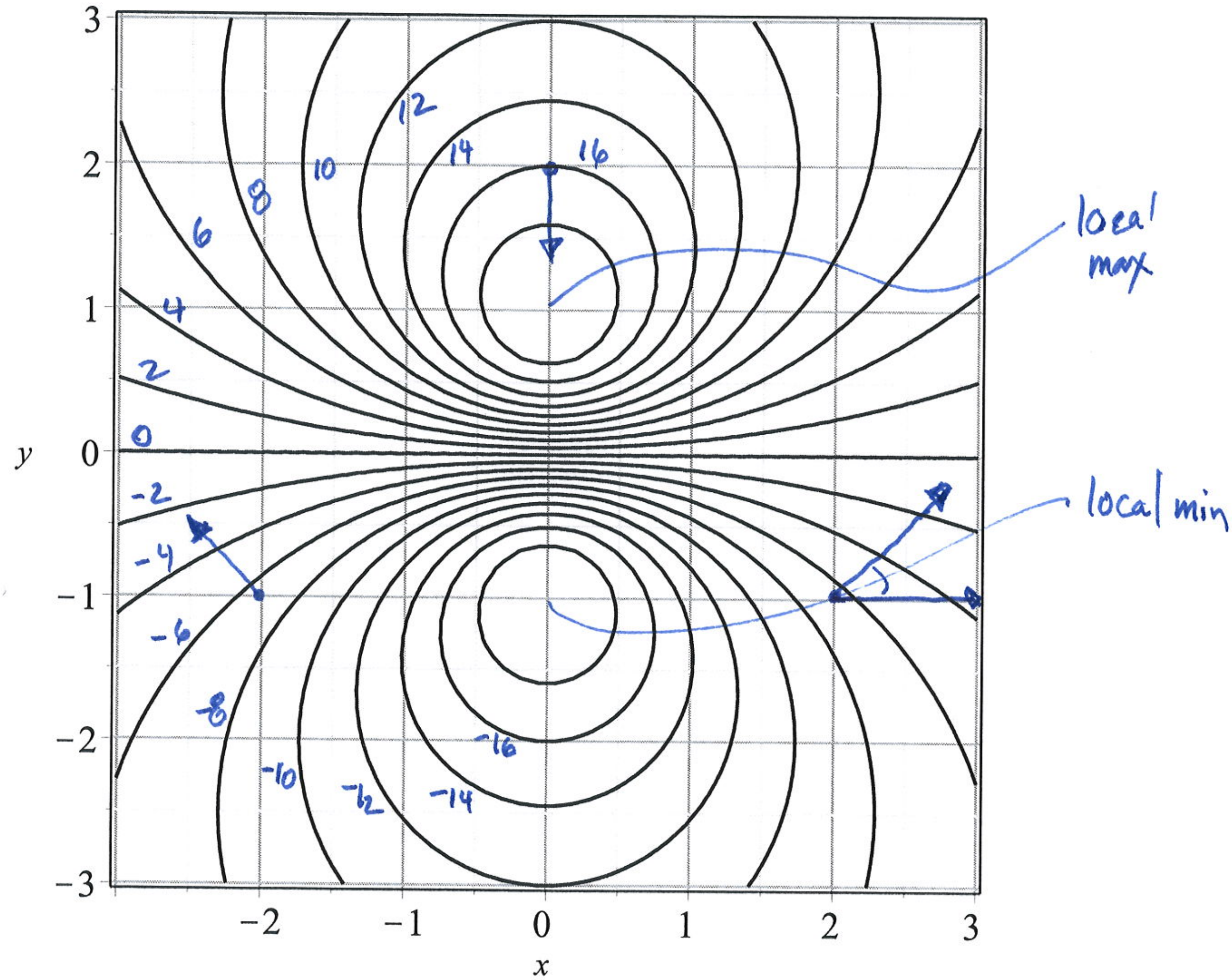
note that contours are tangent to $x^2+y^2=2$ at $(\pm 1, -1)$

$f(x,y) = x^2 + y^2 - y - yx^2$

$= x^2(1-y) + y(y-1) = (y-x^2)(y-1)$

so $f(x,y)=0$ is the line $y=1$ along with $y=x^2$

4. Consider the following contour plot of a function $f(x, y)$:



The values of the contours are spaced evenly. You may make reasonable assumptions about the function: the gradient does not vanish along an entire contours, the function does not fluctuate wildly on a scale smaller than shown by the contours.

The following values are given:

$$f(-3, 0) = 0 \quad f(0, 3) = 12$$

Answer the questions on the following pages.

(a) $\boxed{\text{2 points}}$ Find the unit vector in the direction of the gradient of f at the point $(0, 2)$.

Gradient is perpendicular to contour in the direction of increasing f so $(\vec{\nabla} f)(0,2)$ is in the $-\vec{j}$ direction

Answer: $-\vec{j}$

(b) $\boxed{\text{2 points}}$ Find the unit vector in the direction of the gradient of f at the point $(-2, -1)$.

the direction is $\langle -1, 1 \rangle$ so the unit vector is $\frac{1}{\sqrt{2}}\langle -1, 1 \rangle$

Answer: $\left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

(c) $\boxed{\text{2 points}}$ Find the smallest value of f along the line $y = -2$

min. occurs where contour is tangent to line $y = -2$ which is $(0,-2)$ counting contours we see $f(0,-2) = -16$

Answer: -16

(d) $\boxed{\text{6 points}}$ For each of the following quatities, determine whether the quantity is Positive, Negative, Zero, ~~or "not enough information to determine"~~".

- $f_x(0,1)$

(0,1) is a crit. point and a local max

Answer: Zero

- $f_{xx}(0,1)$

Answer: Negative

- $f_x(1,1)$

Answer: Negative

at (1,1) function decreases as we move to the right but the gaps get bigger so f decreases less rapidly

- $f_{xx}(1,1)$

Answer: Positive

as we move up, the gaps in the x direction gets larger so f_x gets less negative as y increases

- $f_{xy}(1,1)$

Answer: Positive

- $(D_{\vec{\mathbf{i}}}f)(2,-1)$

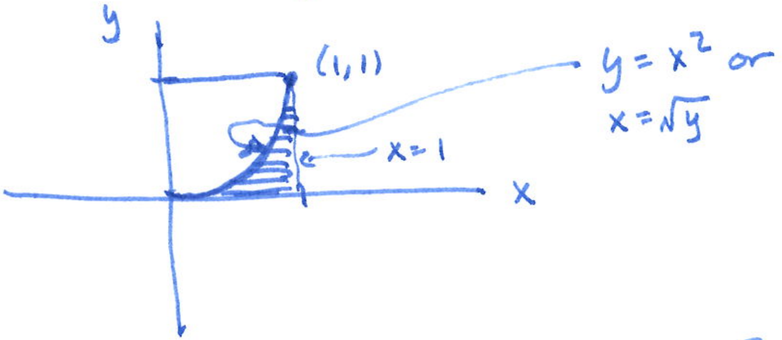
angle between $(\vec{\nabla} f)(2,-1)$ and $\vec{i}$ is ~~obtuse~~ acute

Answer: ~~Negative~~ Positive

5. $\boxed{\text{6 points}}$ Compute the following integral

$$\int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} e^{x^3}\, dx\, dy$$

Need to switch order of integration



$$= \int_{x=0}^{1} \int_{y=0}^{x^2} e^{x^3}\, dy\, dx$$

$$= \int_{x=0}^{1} x^2 e^{x^3}\, dx$$

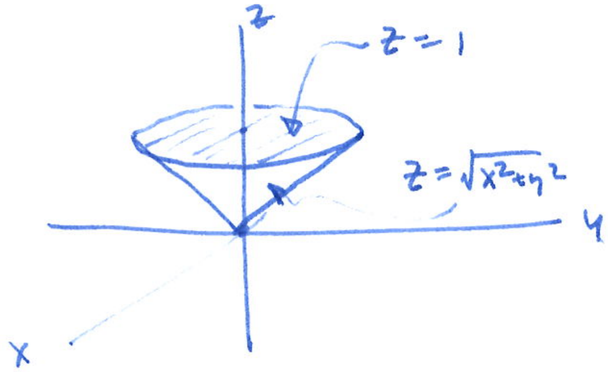
$$= \left[\frac{1}{3} e^{x^3} \right]_{x=0}^{1} = \frac{1}{3}(e-1)$$

Answer:

$\frac{1}{3}(e-1)$

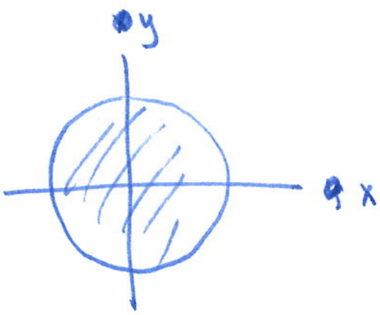
6. Let E be the region above the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 1$.

(a) $\boxed{\text{2 points}}$ Sketch the solid E in the space below:



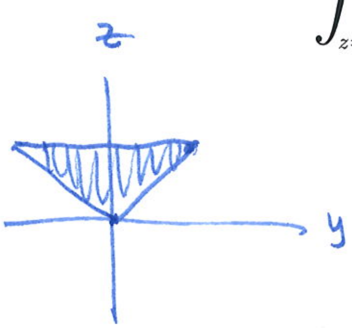
(b) $\boxed{\text{3 points}}$ Fill in the missing limits of integration for $\iiint_E (x^2+y^2)\,d\mathrm{Vol}$ in cartesian coordinates in the given order:

$$\int_{x=-1}^{1}\int_{y=\boxed{-\sqrt{1-x^2}}}^{\boxed{\sqrt{1-x^2}}}\int_{z=\boxed{\sqrt{x^2+y^2}}}^{1}(x^2+y^2)\,dz\,dy\,dx$$



(c) $\boxed{\text{4 points}}$ Fill in the missing limits of integration for $\iiint_E (x^2+y^2)\,d\mathrm{Vol}$ in cartesian coordinates in the given order:

$$\int_{z=0}^{1}\int_{y=\boxed{-z}}^{\boxed{z}}\int_{x=\boxed{-\sqrt{z^2-y^2}}}^{\boxed{\sqrt{z^2-y^2}}}(x^2+y^2)\,dx\,dy\,dz$$



(d) $\boxed{\text{4 points}}$ Fill in the missing limits of integration and integrand for $\iiint_E (x^2 + y^2)\, d\text{Vol}$ in cylindrical coordinates:

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{\boxed{1}} \int_{z=\boxed{r}}^{\boxed{1}} \boxed{r^3}\, dz\, dr\, d\theta$$

(e) $\boxed{\text{3 points}}$ Fill in the missing limits of integration and integrand for $\iiint_E (x^2 + y^2)\, d\text{Vol}$ in spherical coordinates:

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\boxed{\pi/4}} \int_{\rho=0}^{\boxed{\frac{1}{\cos\phi}}} \boxed{\rho^4 \sin^3\phi}\, d\rho\, d\phi\, d\theta$$

$z = 1 \iff \rho\cos\phi = 1 \Rightarrow \rho = \frac{1}{\cos\phi}$

$$x^2 + y^2 = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi = \rho^2\sin^2\phi$$

(f) $\boxed{\text{4 points}}$ Evaluate the integral $\iiint_E (x^2+y^2)d\text{Vol}$ by any method.

Cylindrical:

$$\int_0^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} r^3\,dz\,dr\,d\theta = 2\pi\int_{r=0}^{1} r^3(1-r)\,dr$$

$$= 2\pi\left[\frac{r^4}{4}-\frac{r^5}{5}\right]_0^1 = 2\pi\left[\frac{1}{4}-\frac{1}{5}\right] = \frac{\pi}{10}$$

Spherical:

$$\int_0^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\frac{1}{\cos\phi}} \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\int_{\phi=0}^{\pi/4}\frac{1}{5}\,\frac{\sin^3\phi}{\cos^5\phi}\,d\phi = \frac{2\pi}{5}\int_{\phi=0}^{\pi/4}\left(\frac{1-\cos^2\phi}{\cos^5\phi}\right)\sin\phi\,d\phi$$

$u = \cos\phi$
$du = -\sin\phi$

$$= \frac{2\pi}{5}\int_{u=1}^{\sqrt{2}/2} -\left(\frac{1-u^2}{u^5}\right)du = \frac{2\pi}{5}\int_{\sqrt{2}/2}^{1}(u^{-5}-u^{-3})\,du$$

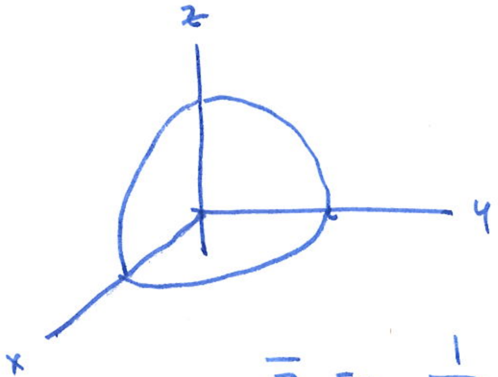
$$= \frac{2\pi}{5}\left[-\frac{1}{4}u^{-4}+\frac{1}{2}u^{-2}\right]_{1/\sqrt{2}}^{1} = \frac{2\pi}{5}\left[-\frac{1}{4}+\frac{1}{2}+\frac{1}{4}(\sqrt{2})^4-\frac{1}{2}(\sqrt{2})^2\right]$$

$$= \frac{2\pi}{5}\left[\frac{-1+2+4-4}{4}\right]$$

$$= \frac{2\pi}{20} = \frac{\pi}{10}$$

Answer: $\pi/10$

7. $\boxed{\text{6 points}}$ Let E be the part of the sphere $x^2 + y^2 + z^2 \leq 1$ which lies in the positive octant. Find the coordinates of the center of mass of E assuming that the density of E is constant. You may use symmetry arguments and the fact that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.



Answer:
$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8}\right)$

by symmetry $\bar{x} = \bar{y} = \bar{z}$

$$\bar{z} = \frac{1}{\mathrm{Vol}(E)} \iiint_E z \, d\mathrm{Vol}.$$

$\mathrm{Vol}(E) = \frac{1}{8} \cdot \frac{4}{3}\pi = \frac{\pi}{6}$ so

$z\,d\mathrm{Vol} = (\rho\cos\phi)\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$$\bar{z} = \frac{6}{\pi} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

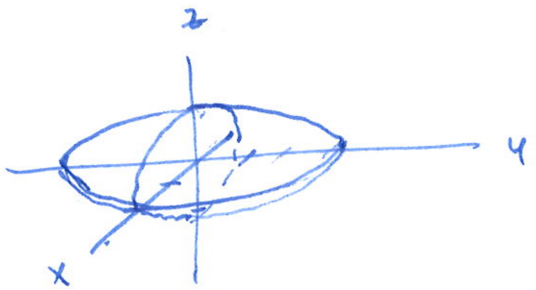
$$= \frac{6}{\pi}\left(\frac{\pi}{2}\right) \int_{\phi=0}^{\pi/2} \cos\phi \sin\phi \left[\frac{\rho^4}{4}\right]_0^1 d\phi$$

$$= \frac{3}{4} \int_{\phi=0}^{\pi/2} \cos\phi \sin\phi = \frac{3}{4}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2} = \frac{3}{8}$$

8. $\boxed{\text{6 points}}$ Find the volume of the largest rectangular box which is contained inside the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1.$$

Answer: $16/\sqrt{3}$



by symmetry the box will be centered at $(0,0,0)$. And its corner in the positive octant will be on the ellipsoid. at a point (x,y,z)

$V(x,y,z) = (2x)(2y)(2z) = 8xyz$ subject to $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

Legrange Multipliers:

$\left(8yz = \frac{2x}{4}\lambda\right)x$

$\left(8xz = \frac{2y}{9}\lambda\right)y$ $\Rightarrow 8xyz = \frac{x^2}{2}\lambda = \frac{2y^2}{9}\lambda = 2z^2\lambda$

$\left(8xy = 2z\lambda\right)z$ so $x^2 = \frac{4y^2}{9} = 4z^2$

$\Rightarrow \frac{x^2}{4} = \frac{y^2}{9} = z^2$ and since these three sum to 1

they must ~~eq~~ each equal $\frac{1}{3}$ i.e. $z = \frac{1}{\sqrt{3}}$ $y = \sqrt{3}$

$x = \frac{2}{\sqrt{3}}$

$V = 8\left(\frac{2}{\sqrt{3}}\right)\left(\sqrt{3}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{16}{\sqrt{3}}$

Extra page for work.