

Final Exam
Math 200/253
June 27th, 2017

Last Name: Solutions

First Name: _____

Student # : _____ Section (200 or 253) : _____

Instructions:

No memory aids allowed. No calculators allowed. No communication devices allowed. Use the space provided on the exam. If you use the back of a page, write "see back" on the front of the page. This exam is 180 minutes long.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	6	
2	10	
3	14	
4	12	
5	6	
6	20	
7	6	
8	6	
Total:	80	

1. 6 points (a) Find the equation of the plane which contains the point $(2, 2, 0)$ and the line

$$x = 5t - 4$$

$$y = -4t + 5$$

$$z = t.$$

The vector $\langle 5, -4, 1 \rangle$ is contained in the plane and since $(t=0)$ $(-4, 5, 0)$ is in the plane as is $(2, 2, 0)$ we have that $\langle 6, -3, 0 \rangle$ is in the plane. Thus we get a normal to the plane:

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -3 & 0 \\ 5 & -4 & 1 \end{vmatrix} = \langle -3, -6, -9 \rangle$$

$$-3(x-2) - 6(y-2) - 9z = 0$$

$$\text{or } (x-2) + 2(y-2) + 3z = 0$$

$$\text{or } x + 2y + 3z = 6$$

Answer:

$$x + 2y + 3z = 6$$

- (b) Find the x -intercept, the y -intercept, and the z -intercept of the plane (namely, the points where the plane hits the x -axis, the y -axis, the z -axis respectively).

get intercepts by setting 2 of the three variables to 0

x intercept: $(6, 0, 0)$

y intercept: $(0, 3, 0)$

z intercept: $(0, 0, 2)$

Answer:

$$(6, 0, 0), (0, 3, 0), (0, 0, 2)$$

2. $f(x, y)$ is a function with

$$f(30, 50) = 40, \quad f_x(30, 50) = 2, \quad f_y(30, 50) = 3$$

- (a) 3 points Find the equation of the plane tangent to the surface $z = f(x, y)$ at the point $(30, 50, 40)$.

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Answer:

$$z = 40 + 2(x - 30) + 3(y - 50)$$

- (b) 3 points Find the approximate value of $f(28, 53)$.

$$\begin{aligned} f(x, y) &\approx 40 + 2(x - 30) + 3(y - 50) \\ &= 40 + 2(28 - 30) + 3(53 - 50) \\ &= 40 - 4 + 9 = 45 \end{aligned}$$

Answer:

$$45$$

- (c) 4 points Let $g(s, t) = f(st, 5s + 4t)$. Compute $g_s(6, 5)$ and $g_t(6, 5)$.

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = 2 \cdot 5 + 3 \cdot 5 = 25 \\ \frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = 2 \cdot 6 + 3 \cdot 4 = 24 \end{aligned}$$

$\frac{\partial x}{\partial s} = t \quad \frac{\partial y}{\partial s} = 5$
 $\frac{\partial x}{\partial t} = s \quad \frac{\partial y}{\partial t} = 4$

Answer:

$$g_s = 25 \quad g_t = 24$$

3. Let

$$f(x, y) = x^2 + y^2 - y - yx^2$$

and let

$$D = \{(x, y) : x^2 + y^2 \leq 2\}$$

(a) 4 points Find all the critical points of $f(x, y)$.

$$\begin{aligned} f_x &= 2x - 2yx = 0 \Rightarrow x=0 \text{ or } 2-2y=0 \Rightarrow y=1 \\ f_y &= 2y-1-x^2=0 \quad \text{if } x=0 \text{ then } 2y-1=0 \Rightarrow y=\frac{1}{2} \\ \text{if } y=1 \text{ then } 1-x^2=0 &\Rightarrow x=\pm 1 \end{aligned}$$

Answer:

$$(0, \frac{1}{2}) (1, 1) (-1, 1)$$

(b) 3 points Classify each of the critical points as a "local maximum", a "local minimum", a "saddle point", or "not an ordinary critical point".

$$\begin{aligned} D &= f_{xx}f_{yy} - f_{xy}^2 = (2-2y) \cdot 2 - (-2x)^2 \\ &= 4(1-y-x^2) \end{aligned}$$

$$D(0, \frac{1}{2}) = 4(1-\frac{1}{2}) = 2 > 0 \quad f_{yy} > 0 \Rightarrow \boxed{(0, \frac{1}{2}) \text{ local min}}$$

$$D(1, 1) = 4(1-1-1) = -4 < 0 \Rightarrow \boxed{(1, 1) \text{ saddle}}$$

$$D(-1, 1) = 4(1-1-1) = -4 < 0 \Rightarrow \boxed{(-1, 1) \text{ saddle}}$$

- (c) 4 points Find the absolute maximum and absolute minimum of $f(x, y)$ on the domain D .

in addition to critical points on the interior, we check
crit. pts on boundary $x^2 + y^2 = 2$

$$\begin{aligned} (2x - 2y^2x = \lambda(2x)) y &\Rightarrow \cancel{2xy} - 2y^2x = \cancel{2xy} - x - x^3 \\ (2y - 1 - x^2 = \lambda(2y)) x &\Rightarrow \downarrow \end{aligned}$$

$$x^2 + y^2 = 2$$

$$x^3 + x - 2y^2x = 0$$

$$\Rightarrow x = 0 \text{ or } x^2 + 1 - 2y^2 = 0$$

if $x = 0$ then $y = \pm\sqrt{2}$
 $(0, \sqrt{2}) \quad (0, -\sqrt{2})$

if $x^2 + 1 - 2y^2 = 0$
 $x^2 + y^2 = 2 \Rightarrow 1 - 3y^2 = -2$
 $3y^2 = 3 \quad y = \pm 1$
 $\Rightarrow x = \pm 1$

boundary:

$$f(0, \sqrt{2}) = 2 - \sqrt{2}$$

$$f(0, -\sqrt{2}) = 2 + \sqrt{2}$$

$$f(1, 1) = 1 + 1 - 1 - 1 = 0$$

$$f(1, -1) = 1 + 1 + 1 + 1 = 4$$

$$f(-1, 1) = 1 + 1 - 1 - 1 = 0$$

$$f(-1, -1) = 1 + 1 + 1 + 1 = 4$$

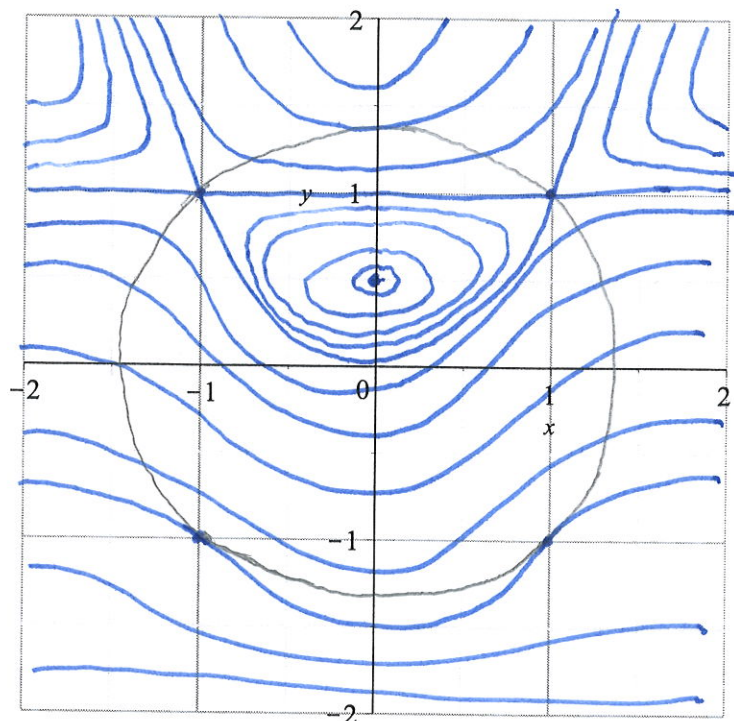
interior:

$$f(0, \frac{1}{2}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \text{ min}$$

maxes

Answer: $(\pm 1, -1)$ maxes
 $(0, \frac{1}{2})$ min.

- (d) 3 points Sketch the contour plot of f on the axes below. Hint: plot the contour $f(x, y) = 0$ by factoring f into a product of two terms.



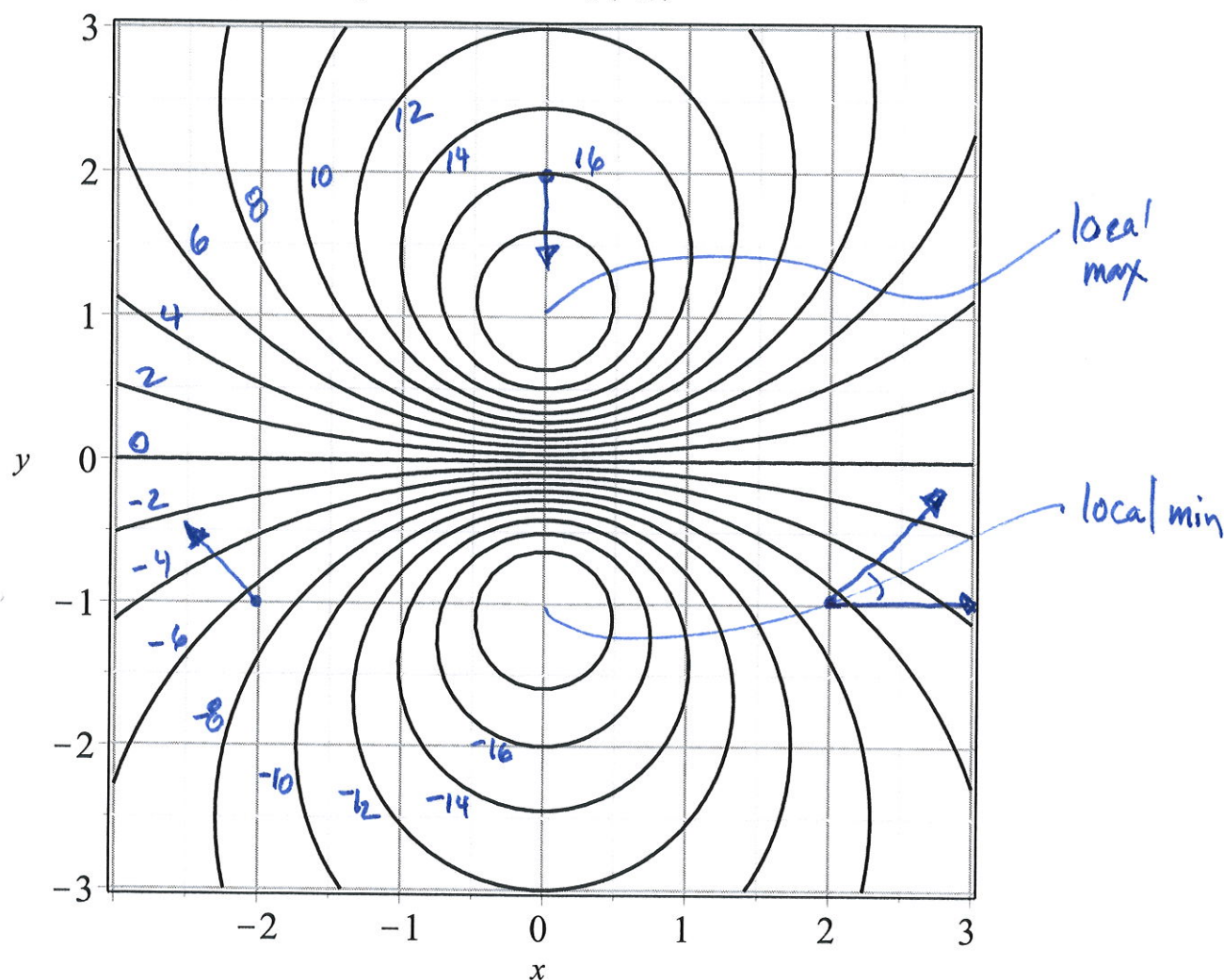
note that
contours are
tangent to
 $x^2 + y^2 = 2$ at
 $(\pm 1, -1)$

$$f(x, y) = x^2 + y^2 - y - yx^2 = \cancel{x^2 + y^2 - y} \cdot \cancel{(y - 1)}$$

$$= x^2(1 - y) + y(y - 1) = (y - x^2)(y - 1)$$

so $f(x, y) = 0$ is the line $y = 1$ along with $y = x^2$

4. Consider the following contour plot of a function $f(x, y)$:



The values of the contours are spaced evenly. You may make reasonable assumptions about the function: the gradient does not vanish along an entire contours, the function does not fluctuate wildly on a scale smaller than shown by the contours.

The following values are given:

$$f(-3, 0) = 0 \quad f(0, 3) = 12$$

Answer the questions on the following pages.

- (a) 2 points Find the unit vector in the direction of the gradient of f at the point $(0, 2)$.

Gradient is perpendicular to contour in the direction of increasing f so $(\vec{\nabla} f)(0, 2)$ is in the $-\vec{j}$ direction

Answer:

$$-\vec{j}$$

- (b) 2 points Find the unit vector in the direction of the gradient of f at the point $(-2, -1)$.

the direction is $\langle -1, 1 \rangle$ so the unit vector is $\frac{1}{\sqrt{2}} \langle -1, 1 \rangle$

Answer:

$$\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

- (c) 2 points Find the smallest value of f along the line $y = -2$

min. occurs where contour is tangent to line $y = -2$
which is $(0, -2)$ counting contours we see
 $f(0, -2) = -16$

Answer:

$$-16$$

- (d) 6 points For each of the following quantities, determine whether the quantity is Positive, Negative, Zero, or ~~"not enough information to determine"~~.

- $f_x(0, 1)$

$(0, 1)$ is a crit. point and a local max

Answer:

Zero

- $f_{xx}(0, 1)$

Answer:

Negative

- $f_x(1, 1)$

Answer:

Negative

- $f_{xx}(1, 1)$

Answer:

Positive

- $f_{xy}(1, 1)$

Answer:

Positive

- $(D_{\vec{i}} f)(2, -1)$

Answer:

~~Negative~~ Positive

at $(1, 1)$ function decreases
as we move to the right
but the gaps get bigger
so f decreases less rapidly

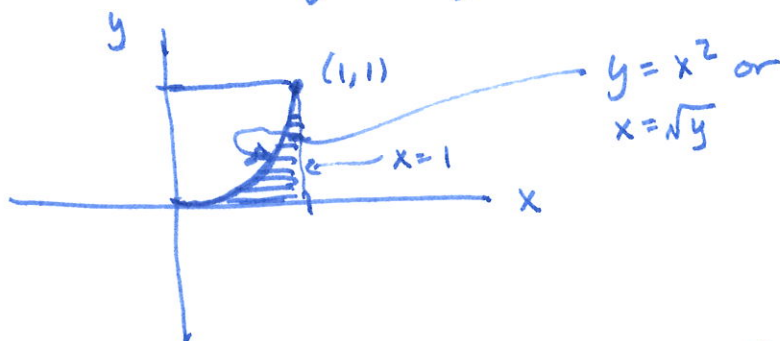
as we move up, the
gaps in the x direction
gets larger so f_x gets
less negative as y increases

angle between $(\vec{\nabla} f)(2, -1)$
and \vec{i} is ~~obtuse~~
acute

5. 6 points Compute the following integral

$$\int_{y=0}^1 \int_{x=\sqrt{y}}^1 e^{x^3} dx dy$$

Need to switch
order of integration



$$\Downarrow$$
$$\int_{x=0}^1 \int_{y=0}^{x^2} e^{x^3} dy dx$$

$$= \int_{x=0}^1 x^2 e^{x^3} dx$$

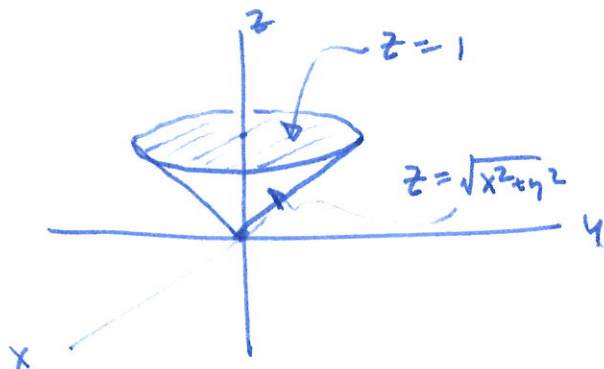
$$= \left[\frac{1}{3} e^{x^3} \right]_{x=0}^1 = \frac{1}{3}(e-1)$$

Answer:

$$\frac{1}{3}(e-1)$$

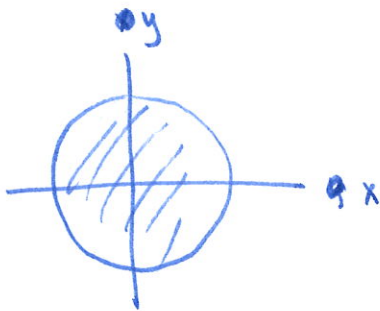
6. Let E be the region above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 1$.

(a) 2 points Sketch the solid E in the space below:



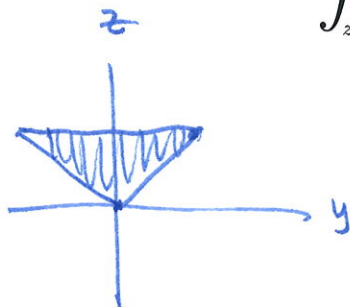
(b) 3 points Fill in the missing limits of integration for $\iiint_E (x^2 + y^2) d\text{Vol}$ in cartesian coordinates in the given order:

$$\int_{x=-1}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=\sqrt{x^2+y^2}}^1 (x^2 + y^2) dz dy dx$$



(c) 4 points Fill in the missing limits of integration for $\iiint_E (x^2 + y^2) d\text{Vol}$ in cartesian coordinates in the given order:

$$\int_{z=0}^1 \int_{y=-z}^z \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (x^2 + y^2) dx dy dz$$



- (d) 4 points Fill in the missing limits of integration and integrand for $\iiint_E (x^2 + y^2) d\text{Vol}$ in cylindrical coordinates:

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{\boxed{1}} \int_{z=\boxed{r}}^{\boxed{1}} \boxed{r^3} dz dr d\theta$$

- (e) 3 points Fill in the missing limits of integration and integrand for $\iiint_E (x^2 + y^2) d\text{Vol}$ in spherical coordinates:

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\boxed{\pi/4}} \int_{\rho=0}^{\boxed{\frac{1}{\cos \phi}}} \boxed{\rho^4 \sin^3 \phi} d\rho d\phi d\theta$$

$$z=1 \Leftrightarrow \rho \cos \phi = 1 \Rightarrow \rho = \frac{1}{\cos \phi}$$

$$x^2 + y^2 = \rho^2 \cos^2 \theta \sin^2 \phi + \rho^2 \sin^2 \theta \sin^2 \phi = \rho^2 \sin^2 \phi$$

(f) 4 points Evaluate the integral $\iiint_E (x^2 + y^2) d\text{Vol}$ by any method.

Cylindrical:
$$\int_0^{2\pi} \int_{r=0}^1 \int_{z=r}^1 r^3 dz dr d\theta = 2\pi \int_{r=0}^1 r^3(1-r) dr$$

$$= 2\pi \left[\frac{r^4}{4} - \frac{r^5}{5} \right]_0^1 = 2\pi \left[\frac{1}{4} - \frac{1}{5} \right] = \frac{\pi}{10}$$

Spherical:
$$\int_0^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\frac{1}{\cos\phi}} \rho^4 \sin^3\phi d\rho d\phi d\theta$$

$$= 2\pi \int_{\phi=0}^{\pi/4} \frac{1}{5} \frac{\sin^3\phi}{\cos^5\phi} d\phi = \frac{2\pi}{5} \int_{\phi=0}^{\pi/4} \left(\frac{1-\cos^2\phi}{\cos^5\phi} \right) \sin\phi d\phi$$

$$= \frac{2\pi}{5} \int_{u=1}^{\sqrt{2}/2} -\left(\frac{1-u^2}{u^5} \right) du = \frac{2\pi}{5} \int_{\sqrt{2}/2}^1 (u^{-5} - u^{-3}) du \quad \left[\begin{array}{l} u = \cos\phi \\ du = -\sin\phi \end{array} \right]$$

$$= \frac{2\pi}{5} \left[-\frac{1}{4} u^{-4} + \frac{1}{2} u^{-2} \right]_{\sqrt{2}/2}^1 = \frac{2\pi}{5} \left[-\frac{1}{4} + \frac{1}{2} + \frac{1}{4} (\sqrt{2})^4 - \frac{1}{2} (\sqrt{2})^2 \right]$$

$$= \frac{2\pi}{5} \left[\frac{-1+2+4-4}{4} \right]$$

$$= \frac{2\pi}{20} = \frac{\pi}{10}$$

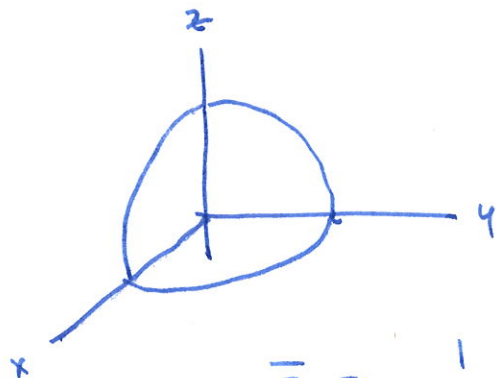
Answer:

$$\frac{\pi}{10}$$

7. [6 points] Let E be the part of the sphere $x^2 + y^2 + z^2 \leq 1$ which lies in the positive octant. Find the coordinates of the center of mass of E assuming that the density of E is constant. You may use symmetry arguments and the fact that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

Answer:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8} \right)$$



by symmetry $\bar{x} = \bar{y} = \bar{z}$

$$\bar{z} = \frac{1}{\text{Vol}(E)} \iiint_E z \, d\text{Vol}.$$

$$\text{Vol}(E) = \frac{1}{8} \cdot \frac{4}{3}\pi = \frac{\pi}{6} \quad \text{so}$$

$$z \, d\text{Vol} = (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\bar{z} = \frac{6}{\pi} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^1 \rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{6}{\pi} \left(\frac{\pi}{2} \right) \int_{\phi=0}^{\pi/2} \cos \phi \sin \phi \left[\frac{\rho^4}{4} \right]_0^1 d\phi$$

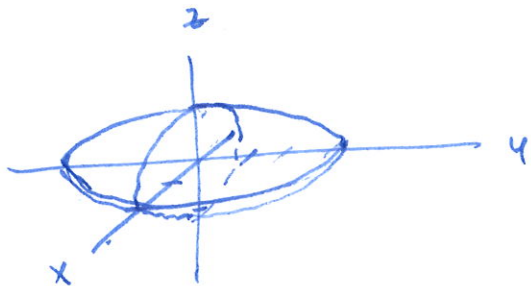
$$= \frac{3}{4} \int_{\phi=0}^{\pi/2} \cos \phi \sin \phi \, d\phi = \frac{3}{4} \left[\frac{1}{2} \sin^2 \phi \right]_0^{\pi/2} = \frac{3}{8}$$

8. 6 points Find the volume of the largest rectangular box which is contained inside the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1.$$

Answer:

$$\frac{16}{\sqrt{3}}$$



by symmetry the box will be centered at $(0,0,0)$. And its corner in the positive octant will be on the ellipsoid at a point (x,y,z)

$$V(x,y,z) = (2x)(2y)(2z) = 8xyz \quad \text{subject to} \quad \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

Lagrange Multipliers:

$$(8yz = \frac{2x}{4}\lambda) \times$$

$$(8xz = \frac{2y}{9}\lambda) y$$

$$(8xy = 2z\lambda) z$$

$$\Rightarrow 8xyz = \frac{x^2}{2}\lambda = \frac{2y^2}{9}\lambda = 2z^2\lambda$$

$$\text{so } x^2 = \frac{4y^2}{9} = 4z^2$$

$$\Rightarrow \frac{x^2}{4} = \frac{y^2}{9} = z^2 \quad \text{and since these three sum to 1}$$

they must each equal $\frac{1}{3}$ i.e. $z = \frac{1}{\sqrt{3}} \quad y = \sqrt{3}$

$$x = \frac{2}{\sqrt{3}}$$

$$V = 8\left(\frac{2}{\sqrt{3}}\right)(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right) = \frac{16}{\sqrt{3}}$$

Extra page for work.