(1) Consider the integral \( \iint_R f(x, y) \, dA \) where \( R \) is the bounded region enclosed above the graph of the function \( y = |x^2 - 1| \) and below the parabola \( y = 5 - x^2 \).
(a) Sketch the region of integration \( R \) in the \( x-y \) plane.
(b) Write the integral as an iterated integral in the order \( dx \, dy \), then repeat in the order \( dy \, dx \).

\[
\iint_R f(x, y) \, dA = \int_{-\sqrt{1+y}}^{\sqrt{1-y}} \int_0^1 f(x, y) \, dx \, dy + \int_{\sqrt{1+y}}^{\sqrt{1-y}} \int_0^1 f(x, y) \, dx \, dy + \int_{-\sqrt{1+y}}^{\sqrt{1-y}} \int_{\sqrt{1-y}}^{\sqrt{1+y}} f(x, y) \, dx \, dy + \int_{\sqrt{1-y}}^{\sqrt{1+y}} \int_0^{\sqrt{5-y}} f(x, y) \, dx \, dy
\]

\[
\iint_R f(x, y) \, dA = \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2-1}^{5-x^2} f(x, y) \, dx \, dy + \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2-1}^{1-x^2} f(x, y) \, dx \, dy + \int_{-\sqrt{5-y}}^{\sqrt{5-y}} \int_{-\sqrt{5-y}}^{\sqrt{5-y}} f(x, y) \, dx \, dy
\]

(2) Use polar coordinates to find the area of the region which is the intersection of the cardioid \( r = 1 + \cos \theta \) and the circle \( x^2 + y^2 = 9/4 \).

\[
\iint_R f(x, y) \, dA = \int_{-\pi/3}^{\pi/3} \int_0^{1+\cos \theta} f(r, \theta) \, r \, dr \, d\theta
\]

Solution.
Note that the intersection between the graphs happens when
\( 1 + \cos \theta = 3/2 \iff \theta = \pm \pi/3 \)
Let us denote the region by $R$, and notice that it is symmetric with respect to the $x$-axis. Hence it’s enough to calculate the area of the region located above the $x$-axis and multiply the result by 2.

\[
\text{Area} (R) = \int \int_{R} dA = 2 \left( \int_{0}^{\pi/3} \int_{0}^{\pi/2} r\,dr\,d\theta + \int_{\pi/3}^{\pi} \int_{0}^{3/2} r\,dr\,d\theta \right)
\]

\[
\int_{0}^{\pi/3} \int_{0}^{\pi/2} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \left[ \frac{r^2}{2} \right]_{r=0}^{r=\frac{3}{2}} d\theta = \frac{9}{8} \cdot \frac{2\pi}{3} = \frac{3\pi}{4}
\]

\[
\int_{\pi/3}^{\pi} \int_{0}^{\pi/2} r\,dr\,d\theta = \int_{\pi/3}^{\pi} \left[ \frac{r^2}{2} \right]_{r=0}^{r=\frac{3}{2}} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_{\pi/3}^{\pi} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \left( \frac{\pi}{3} - \frac{9}{8} \sqrt{3} \right)
\]

We get $\text{Area} (R) = \frac{5}{2}\pi - \frac{9}{8} \sqrt{3}$.

(3) Calculate the integral $\int_{0}^{16} \int_{0}^{y^{1/4}} \frac{1}{3 + x^5} \,dx\,dy$.

**Solution.** Changing the order of integration we get

\[
\int_{0}^{16} \int_{0}^{y^{1/4}} \frac{1}{3 + x^5} \,dx\,dy = \int_{0}^{2} \int_{0}^{x^4} \frac{1}{3 + x^5} \,dy\,dx = \int_{0}^{2} \frac{x^4}{3 + x^5} \,dx = \frac{1}{5} \left[ \log (3 + x^5) \right]_{x=0}^{x=2} = \frac{1}{5} (\log 35 - \log 3)
\]