(1) Consider the surface described by the equation
\[ e^{3-x^2-y^2-z^2} + 1 = 2xyz \]

(a) Find a function \( F(x, y, z) \) such that \((a, b, c)\) is on the surface if and only if we have \( F(a, b, c) = 0 \). Compute the gradient of \( F \).

(b) Give the equation for the tangent plane to the surface at the point \( p = (1, 1, 1) \).

Solution.

(a) \( F(x, y, z) = e^{3-x^2-y^2-z^2} + 1 - 2xyz \),
\[ \nabla F(x, y, z) = \langle -2xe^{3-x^2-y^2-z^2} - 2yz, -2ye^{3-x^2-y^2-z^2} - 2xz, -2ze^{3-x^2-y^2-z^2} - 2xy \rangle \]

(b) \( \nabla F(1, 1, 1) = \langle -4, -4, -4 \rangle \) hence the tangent plane to the surface at point \( p = (1, 1, 1) \) is
\[ -4(x - 1) - 4(y - 1) - 4(z - 1) = 0 \text{ or } x + y + z = 3 \]

(2) The temperature in space is given by the function
\[ T(x, y, z) = x^2e^y - xy^2 + z \sqrt{x} \]

(a) What is the rate of change of temperature at point \( P = (1, 1, 1) \) in the direction of the vector \( \vec{u} = (2, 1, 1) \).

(b) You are feeling a little cold when standing at point \( P \). In which direction should you be going in order to increase the temperature in the fastest possible way?

(c) Suppose you are warm enough at point \( P \), and you want to walk keeping the temperature constant. In which direction should you be going?

Solution. \( \nabla T(x, y, z) = \langle 2xe^y - y^2 + \frac{z}{2 \sqrt{x}}, x^2e^y - 2xy, \sqrt{x} \rangle \).

(a) We're looking for \( D_\vec{u}T(P) \) where \( \vec{u} = \frac{1}{\sqrt{6}} \vec{v} \) is a direction vector in the direction of \( \vec{v} \), hence
\[ D_\vec{u}T(P) = \nabla TP \cdot \vec{u} = \left\langle 2e - \frac{1}{2}, e - 2, 1 \right\rangle \cdot \frac{1}{\sqrt{6}} (2, 1, 1) = \frac{5e - 2}{\sqrt{6}} \]

(b) You need to walk in the direction of the gradient, that is in the direction of the vector \( \vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} \), where
\[ \vec{v} = \left\langle 2e - \frac{1}{2}, e - 2, 1 \right\rangle \]

(c) You can walk in the direction of any vector contained in the tangent plane to the level surface of \( T(x, y, z) = T(P) \) at point \( P \). Recall that vectors in the tangent plane are perpendicular to \( \nabla T(P) \). Suppose \( Q = (x, y, z) \) is on the tangent plane hence we would like \( \overrightarrow{PQ} \cdot \nabla T(P) = 0 \) or
\[ (x - 1, y - 1, z - 1) \cdot (2e - 1/2, e - 2, 1) = 0 \]
We get
\[(x - 1) (2e - \frac{1}{2}) + (y - 1) (e - 2) + z - 1 = 0\]
For convenience we can pick, for instance, \(x = 0, y = 3\) and this gives \(z = \frac{9}{2}\).
Now we need to go in the direction of \(\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}\), where
\[\vec{v} = \langle -1, 2, \frac{7}{2} \rangle.\]

**Remark.** There are many other possible directions.

(3) Find and classify all critical points of \(f(x, y) = x^3 + 2y^3 - 3x^2 - 24y + 6\).

**Solution.** A critical point is for which \(\nabla f(x, y) = 0\)

\[
\begin{align*}
    f_x(x, y) &= 3x^2 - 6x = 0 \\
    f_y(x, y) &= 6y^2 - 24 = 0
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
3x(x - 2) = 0 \\
y^2 = 4
\end{cases} \Rightarrow x = 0, 2 \text{ and } y = \pm 2
\]

we get 4 different points \(p_1 = (0, 2), p_2 = (0, -2), p_3 = (2, 2), p_4 = (2, -2)\). The discriminant is given by

\[D(x, y) = (6x - 6) \cdot 12y\]

then

<table>
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<tr>
<th>point</th>
<th>(D(p))</th>
<th>(f_{xx}(p))</th>
<th>type</th>
</tr>
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