(1) Consider the function \( f(x, y) = x^3 \log y + \frac{y}{x^2} \) and the point \( P = (1, 1) \).
(a) Find all first order partial derivatives of \( f \) as well as \( f_{xy}, f_{yx}, \) and calculate their value at \( P \).
(b) Use the linear approximation of \( f \) at the point \( P \) to approximate \( f(0.95, 1.02) \).
(c) Write the equation of the tangent plane to the graph of \( f(x, y) \) at \((1, 1, 1)\).
(d) Suppose that the tangent plane to the graph of \( f(x, y) \) at \( Q = (a, e, c) \) has the equation 
\[ z = 4e - 3 + (3 - 2e) x + (\frac{1}{e} - 1) y. \] Find \( a, c). \)

Remark. Note that \( c \) is not a variable, but the number which satisfy the equation \( \log e = 1 \).

Solution.
(a) \( f_x(x, y) = 3x^2 \log y - \frac{2y}{x^3} \), \( f_y(x, y) = \frac{x^3}{y} + \frac{1}{x^2} \) and \( f_{xy}(x, y) = f_{yx}(x, y) = \frac{3x^2}{y} - \frac{2}{x^2} \).
Hence \( f_x(1, 1) = -1, f_y(1, 1) = 2, f_{xy}(1, 1) = f_{yx}(1, 1) = 1 \).
(b) The linear approximation for \( f(x, y) \) near \( P \) is given by the equation
\[ L(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 1 - (x - 1) + 2(y - 1) = 2y - x \]

hence
\[ f(0.95, 1.02) \sim L(0.95, 1.02) = 2 \cdot 0.95 - 1.02 = 0.88 \]
(c) This is exactly \( z = L(x, y) \).
(d) Since the point \( Q \) is on the place we have
\[ c = 4e - 3 + (3 - 2e)a + (\frac{1}{e} - 1)e \Rightarrow c = 3e - 2 + (3 - 2e)a \]
Since it’s on the graph of \( f \) we have
\[ c = f(a, e) = a^3 + \frac{e}{a^2} \]
therefore \( a = 1 \) satisfies both equations with \( c = 1 + e \).

(2) In an alternate universe the magnetic force in space is described by the function \( F(x, y, z) = \frac{xy}{y^2 + x^2} \). Your position at time \( t \) is space is \((x(t), y(t), z(t))\) where \( x(0) = 1, y(0) = 1, z(0) = 1 \) and \( x'(0) = 3, y'(0) = 2, z'(0) = 4 \). What rate of change of the magnetic force do you experience at \( t = 0 \)?

Solution. We need to find \( \frac{dF}{dt}(0) \). Since \((x(0), y(0), z(0)) = (1, 1, 1)\), by the chain rule this is given by
\[ \frac{dF}{dt}(0) = F_x(1, 1, 1) \cdot x'(0) + F_y(1, 1, 1) \cdot y'(0) + F_z(1, 1, 1) \cdot z'(0) \]

Now
\[
\begin{align*}
F_x(x, y, z) &= \frac{x(y^2 + x^2) - xyz}{(y^2 + x^2)^2} \\
F_y(x, y, z) &= \frac{y(y^2 + x^2) - 2yz}{(y^2 + x^2)^2} \\
F_z(x, y, z) &= -\frac{xy(y^2 + x^2)}{(y^2 + x^2)^2}
\end{align*}
\]

\[
\begin{align*}
F_x(1, 1, 1) &= \frac{1}{4} \\
F_y(1, 1, 1) &= 0 \\
F_z(1, 1, 1) &= -\frac{1}{2}
\end{align*}
\]

hence
\[ \frac{dF}{dt}(0) = \frac{1}{4} \cdot 3 + 0 \cdot 2 - \frac{1}{2} \cdot 4 = -\frac{5}{4} \]