SPLITTING FOR INTEGER TILINGS AND THE COVEN-MEYEROWITZ TILING CONDITIONS

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Abstract. Suppose that a finite set $A \subset \mathbb{Z}$ tiles the integers by translations. By periodicity, any such tiling is equivalent to a factorization $A \oplus B = \mathbb{Z}_M$ of a finite cyclic group. We are interested in investigating the structure of such tilings, and in particular in a tentative characterization of finite tiles proposed by Coven and Meyerowitz [2]. Building on the work in [24, 25], we prove that the Coven-Meyerowitz condition (T2) holds for all integer tilings of period $M = (p_i p_j p_k)^2$, where p_i, p_j, p_k are distinct primes. This extends the main result of [25], where we assumed that M is odd. We also provide a new combinatorial interpretation of some of the main tools of [24], [25], and introduce a new method based on *splitting*. In addition to applications in the proof of (T2) in the even case, we use splitting to prove weaker structural results in cases where (T2) is not yet available.

Contents

Date: July 24, 2022.
2020 *Mathematics Subject Classification.* Primary 05B45, 11B75, 20K01. Secondary 11C08, 43A47, 51D20, 52C22.

1. Introduction

A finite set $A \subset \mathbb{Z}$ *tiles the integers by translations* if there is a (necessarily infinite) set $T \subset \mathbb{Z}$ such that every $n \in \mathbb{Z}$ can be represented uniquely as $n = a + t$ with $a \in A$ and $t \in T$. We will call such sets *finite tiles*. Newman [35] proved[1] that any tiling of $\mathbb{Z}$ by a finite set A must be periodic: there exists a $M \in \mathbb{N}$ such that $T = B \oplus M\mathbb{Z}$ for some finite set $B \subset \mathbb{Z}$. We then have $|A|\,|B| = M$, and $A \oplus B$ modulo M is a factorization of the cyclic group $\mathbb{Z}_M$. We will write this as $A \oplus B = \mathbb{Z}_M$.

This article is a continuation of our program of study of integer tilings, initiated in [24, 25]. We develop new methods to investigate such tilings, extend the range of tilings for which the Coven-Meyerowitz tiling conditions [2] must hold, and prove partial structural results in certain situations where the Coven-Meyerowitz characterization is currently unavailable. We also improve large parts of the argument from [25].

[1] A better quantitative estimate on the period of the tiling is due to Greenfeld and Tao [13].

The Coven-Meyerowitz conditions are stated in the language of cyclotomic polynomials, which we now introduce. By translational invariance, we may assume that $A, B \subset \{0, 1, \dots\}$ and that $0 \in A \cap B$. The *characteristic polynomials* (also known as *mask polynomials*) of A and B are

$$A(X) = \sum_{a \in A} X^a, \;\; B(x) = \sum_{b \in B} X^b.$$

The tiling condition $A \oplus B = \mathbb{Z}_M$ is then equivalent to

$$A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1). \tag{1.1}$$

Let $\Phi_s(X)$ be the s-th cyclotomic polynomial, i.e., the unique monic, irreducible polynomial whose roots are the primitive s-th roots of unity. Alternatively, Φ_s allow the inductive definition

$$X^n - 1 = \prod_{s|n} \Phi_s(X). \tag{1.2}$$

In particular, (1.1) is equivalent to

$$|A||B| = M \text{ and } \Phi_s(X) \mid A(X)B(X) \text{ for all } s|M, \; s \neq 1. \tag{1.3}$$

Since Φ_s are irreducible, each $\Phi_s(X)$ with $s|M$ must divide at least one of $A(X)$ and $B(X)$.

Coven and Meyerowitz [2] proved the following theorem.

Theorem 1.1. [2] *Let S_A be the set of prime powers p^α such that $\Phi_{p^\alpha}(X)$ divides $A(X)$. Consider the following conditions.*

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$,

(T2) if $s_1, \dots, s_k \in S_A$ are powers of different primes, then $\Phi_{s_1 \dots s_k}(X)$ divides $A(X)$.

Then:

- *if A satisfies (T1), (T2), then A tiles $\mathbb{Z}$;*
- *if A tiles $\mathbb{Z}$ then (T1) holds;*
- *if A tiles $\mathbb{Z}$ and $|A|$ has at most two distinct prime factors, then (T2) holds.*

Part (T1) is a counting condition, ensuring that the factors A and B in any tiling $A \oplus B = \mathbb{Z}_M$ satisfy $|A||B| = M$. The second condition (T2) is a much deeper structural property. For finite sets A satisfying (T1) and (T2), Coven and Meyerowitz constructed an explicit tiling $A \oplus B^\flat = \mathbb{Z}_M$, where $M = \text{lcm}(S_A)$ and $B^\flat$ is an explicit "standard" tiling complement (described here in Section 2.4). We proved in [24] (although this argument was already implicit in [2]) that having a tiling complement of this type is in fact equivalent to (T2). This places (T2) in close relation to questions on factor replacement in factorizations of abelian groups [45].

The Coven-Meyerowitz proof can be extended to a limited range of tilings where M has more than two prime factors. In [24, Corollary 6.2], we use the methods of [2] to prove that if $A \oplus B = \mathbb{Z}_M$, and if $|A|$ and $|B|$ have at most two *shared* distinct prime factors, then both A and B satisfy (T2). (Similar results have also appeared elsewhere in the literature, see e.g., [47], [40, Proposition 4.1], [31, Theorem 1.5].)

Unfortunately, the methods of [2] do not extend to the case when $|A|$ and $|B|$ share three or more distinct prime factors. This was already known to Coven and Meyerowitz, who cited the examples due to Szabó [44] (see also [26]). While (T2) still holds for these particular examples, a key element of the Coven-Meyerowitz proof is Sands's factor replacement theorem [38], which no longer holds in this setting.

In [25], we proved the following theorem.

Theorem 1.2. [25] *Let $M = p_i^2 p_j^2 p_k^2$, where p_i, p_j, p_k are distinct odd primes. Assume that $A \oplus B = \mathbb{Z}_M$, with $|A| = |B| = p_i p_j p_k$. Then both A and B satisfy (T2).*

One of the main goals of this article is to extend Theorem 1.2 to the even case, as follows.

Theorem 1.3. *Let $M = p_i^2 p_j^2 p_k^2$, where p_i, p_j, p_k are distinct primes and $2 \in \{p_i, p_j, p_k\}$. Assume that $A \oplus B = \mathbb{Z}_M$, with $|A| = |B| = p_i p_j p_k$. Then both A and B satisfy (T2).*

Theorems 1.2 and 1.3 cover all tilings of period $M = p_i^2 p_j^2 p_k^2$, where p_i, p_j, p_k are distinct primes. Additionally, the results of this article together with those of [25] provide a classification of all tilings $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. The detailed statements are given in Theorems 6.1 and 6.2.

Our second goal is to develop new methods in the study of integer tilings, and improve those introduced in [24, 25].

One of the new concepts defined in [24] is the *box product*, based on an identity proved in [11] and reproduced here as Theorem 3.8. The box product, and related methods such as saturating sets, are crucial parts of our proof of (T2) for tilings with 3 prime factors. In Section 3, we give a new combinatorial interpretation of the box product, and a new proof of Theorem 3.8, based on the theorems of Sands and Tijdeman on the dilation invariance of tiling properties of sets. This brings it closer to the dilation-based approach of [13]. It also allows a similar reinterpretation of saturating sets (see Lemma 4.6).

The main new method introduced in this paper is *splitting* (Section 4). The idea behind it is quite elementary: given an M-fiber in $\mathbb{Z}_M$ (that is, an arithmetic progression of step M/p and length p for some prime divisor $p|M$), we ask which elements of A and B tile the elements of that progression. It is easy to prove that, for each fiber separately, these elements must follow a certain "splitting" pattern.

Splitting is closely related to saturating sets. Our dilation-based reinterpretation of the latter shows them to be the sets of elements of A and B that tile a fixed element z of $\mathbb{Z}_M$ in tilings $A \oplus rB$, where r ranges over an appropriate set of dilations. Splitting does not take dilations into account; however, instead of tiling just one element of $\mathbb{Z}_M$, we tile an entire arithmetic progression at the same time. This turns out to be more efficient than saturating set techniques in some situations, particularly in the even case (see Sections 9.2 and 12.1 for examples).

The concept of splitting gives rise to many natural and interesting questions. For example, one can ask whether splitting patterns are uniform for all M-fibers in the same direction (i.e. corresponding to the same prime factor p), or whether this uniformity persists under dilations of the sets A and B. This turns out to have close connections both to the *slab reduction* defined in [24, Section 6.2] and to the splitting properties of box products discussed in [25, Section 9.2]. We make these connections explicit in Section 5.2. The splitting interpretation of the slab reduction also makes it easier to formulate partial results in this direction, such as local uniformity restricted to certain grids. An example of this is given in Section 11.

The third purpose of this paper is to improve and streamline the fibered grids argument in [25, Section 9]. This was the most technical part of our proof of (T2) in the 3-prime odd case. We revisit it here in Sections 10–13. In addition to providing the arguments needed to cover the even case, we reorganize the proof of our main intermediate result (Theorem 10.1, part (II a)) in a way that seems to capture the phenomena at hand much better. Further simplifications are due to the use of the splitting formulations of the slab reduction in Section 13. We also strengthen some of our technical arguments. We include both odd and even cases in this part of the article, given that the improvements apply in both cases and the additional length needed to cover the odd case is minimal.

We will rely on the methods and concepts introduced in [24, 25]. In order to keep this article reasonably readable, we include a brief summary of the necessary results from [24, 25] (see e.g., Sections 2 and 7). A reader willing to accept and use those as black boxes should be able to read most of the paper without consulting [24, 25] at every step. We do refer to [24, 25] for some of the more specialized technical arguments, such as the odd case of Corollary 10.8 (ii) or Claims 6-8 in the proof of Proposition 12.10.

There is relatively little prior work on (T2). Since [2], and prior to our work in [24, 25], progress in this direction has been limited to special cases that either assume a particular structure of the tiling (see [22], [4]) or are covered by the methods of [2] (see [47], [40], [31]). However, there is a considerable body of work on other tiling questions, for example tilings of $\mathbb{Z}^2$ ([1], [13]) and tilings of the real line by a function (see [19] for a survey).

There is also significant interest in a closely related conjecture due to Fuglede [10] which states that a set $\Omega \subset \mathbb{R}^n$ of positive n-dimensional Lebesgue measure tiles $\mathbb{R}^n$ by translations if and only if the space $L^2(\Omega)$ admits an orthogonal basis of exponential functions. A set with the latter property is called *spectral*. The conjecture is known to be false, in its full generality, in dimensions 3 and higher [46], [20], [21], [8], [34], [9]. However, there are important special cases in which the conjecture was confirmed [14], [12], [30], and the finite abelian group analogue of the conjecture is currently a very active area of research [15], [32], [7], [16], [17], [18], [31], [40], [41], [42], [6], [49].

In dimension 1, the problem is still open in both directions, and the "tiling implies spectrum" direction hinges on proving (T2) for all finite tiles ([27], [28], [23]; see also [5] for an overview of the problem and an investigation of the converse direction). Our Theorems 1.2 and 1.3, combined with [23, Theorem 1.5] and [24, Corollary 6.2], extend Corollary 1.3 in [25] to the even case.

Corollary 1.4. *Let* $M = p_i^2 p_j^2 p_k^2$.

(i) *If* $A \subset \mathbb{Z}_M$ *tiles* $\mathbb{Z}_M$ *by translations, then it is spectral.*
(ii) *Let* $A \subset \mathbb{Z}$ *be a finite set such that* A *mod* M *tiles* $\mathbb{Z}_M$*, and let* $F = \bigcup_{a\in A}[a, a+1]$*, so that* F *tiles* $\mathbb{R}$ *by translations. Then* F *is spectral.*

2. Notation and preliminaries

This section summarizes the relevant definitions and results of [24], specialized to the 3-prime case. All material due to other authors is indicated explicitly as such.

2.1. **Multisets and mask polynomials.** We will assume that $M = p_1^{n_1} \dots p_K^{n_K}$, where $p_1, \dots, p_K$ are distinct primes and $n_1, \dots, n_K \in \mathbb{N}$. The results of Sections 3, 4, and 5 are not restricted to the 3-prime setting, hence in that part of the article we allow K to be arbitrary.

In the 3-prime setting, we will write $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, where p_i, p_j, p_k are distinct primes and $n_i, n_j, n_k \in \mathbb{N}$. While the indices $\{i, j, k\}$ are a permutation of $\{1, 2, 3\}$, we will use i, j, k for this purpose, freeing up numerical subscripts for other uses. The full proof of Theorem 1.2 requires that $n_i = n_j = n_k = 2$ and $2 \in \{p_i, p_j, p_k\}$. However, many of our intermediate results are valid under weaker assumptions as indicated.

We will always work in either $\mathbb{Z}_M$ or in $\mathbb{Z}_N$ for some $N|M$. We use $A(X)$, $B(X)$, etc. to denote polynomials modulo $X^M - 1$ with integer coefficients. Each such polynomial $A(X) = \sum_{a \in \mathbb{Z}_M} w_A(a) X^a$ is associated with a weighted multiset in $\mathbb{Z}_M$, which we will also denote by A, with weights $w_A(x)$ assigned to each $x \in \mathbb{Z}_M$. (If the coefficient of X^x in $A(X)$ is 0, we set $w_A(x) = 0$.) In particular, if A has $\{0, 1\}$ coefficients, then w_A is the characteristic function of a set $A \subset \mathbb{Z}_M$. We will use $\mathcal{M}(\mathbb{Z}_M)$ to denote the family of all weighted multisets in $\mathbb{Z}_M$, and reserve the notation $A \subset \mathbb{Z}_M$ for sets.

If $N|M$, then any $A \in \mathcal{M}(\mathbb{Z}_M)$ induces a weighted multiset $A \bmod N$ in $\mathbb{Z}_N$, with the corresponding mask polynomial $A(X) \bmod (X^N - 1)$ and induced weights

$$w_A^N(x) = \sum_{x' \in \mathbb{Z}_M : x' \equiv x \bmod N} w_A(x'), \quad x \in \mathbb{Z}_N. \tag{2.1}$$

We will continue to write A and $A(X)$ for $A \bmod N$ and $A(X) \bmod X^N - 1$, respectively, while working in $\mathbb{Z}_N$.

If $A, B \in \mathcal{M}(\mathbb{Z}_M)$, we will use $A + B$ to indicate the weighted multiset corresponding to the mask polynomial $(A + B)(X) = A(X) + B(X)$, with the weight function $w_{A+B}(x) = w_A(x) + w_B(x)$. We use the convolution notation $A * B$ to denote the weighted sumset of A and B, so that $(A * B)(X) = A(X)B(X)$ and

$$w_{A*B}(x) = (w_A * w_B)(x) = \sum_{y \in \mathbb{Z}_M} w_A(x - y) w_B(y).$$

If one of the sets is a singleton, say $A = \{x\}$, we will simplify the notation and write $x * B = \{x\} * B$. The direct sum notation $A \oplus B$ is reserved for tilings, i.e., $A \oplus B = \mathbb{Z}_M$ means that $A, B \subset \mathbb{Z}_M$ are both sets and $A(X)B(X) = \frac{X^M - 1}{X - 1} \bmod X^M - 1$. We will not use derivatives of polynomials in this paper, hence notation such as A', A'', etc., will be used to denote auxiliary multisets and polynomials rather than derivatives.

2.2. **Array coordinates and geometric representation.** By the Chinese Remainder Theorem, the cyclic group $\mathbb{Z}_M$ may be identified with $\mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \ldots \oplus \mathbb{Z}_{p_K^{n_K}}$. We set up an explicit isomorphism as follows. For $\nu \in \{1, \ldots, K\}$, define $M_\nu := M/p_\nu^{n_\nu} = \prod_{\kappa \neq \nu} p_\kappa^{n_\kappa}$. Then each $x \in \mathbb{Z}_M$ can be written uniquely as

$$x = \sum_{\nu=1}^{K} \pi_\nu(x) M_\nu, \quad \pi_\nu(x) \in \mathbb{Z}_{p_\nu^{n_\nu}}.$$

Geometrically, this maps each $x \in \mathbb{Z}_M$ to an element of a K-dimensional lattice with coordinates $(\pi_1(x), \ldots, \pi_K(x))$. The tiling $A \oplus B = \mathbb{Z}_M$ corresponds to a tiling of that lattice.

Let $D|M$ and $\nu \in \{1, \ldots, K\}$. A *D-grid* in $\mathbb{Z}_M$ is a set of the form

$$\Lambda(x, D) := x + D\mathbb{Z}_M = \{x' \in \mathbb{Z}_M : \; D|(x - x')\}$$

for some $x \in \mathbb{Z}_M$. We note the following important special cases.

- A *line* through $x \in \mathbb{Z}_M$ in the p_ν direction is the set $\ell_\nu(x) := \Lambda(x, M_\nu)$.
- A *plane* through $x \in \mathbb{Z}_M$ perpendicular to the p_ν direction, on the scale $M_\nu p_\nu^{\alpha_\nu}$, is the set $\Pi(x, p_\nu^{\alpha_\nu}) := \Lambda(x, p_\nu^{\alpha_\nu})$.
- An *M-fiber in the p_ν direction* is a set of the form $x * F_\nu$, where $x \in \mathbb{Z}_M$ and

$$F_\nu = \{0, M/p_\nu, 2M/p_\nu, \dots, (p_\nu - 1)M/p_\nu\}. \tag{2.2}$$

Thus $x * F_\nu = \Lambda(x, M/p_\nu)$.

We also need more general fibers, defined as follows. Let $N|M$, $c \in \mathbb{N}$, and $\nu \in \{i, j, k\}$ such that $p_\nu|N$. An *N-fiber in the p_ν direction* with multiplicity c is a set $F \subset \mathbb{Z}_M$ such that F mod N has the mask polynomial

$$F(X) \equiv cX^a(1 + X^{N/p_\nu} + X^{2N/p_\nu} + \dots + X^{(p_\nu-1)N/p_\nu}) \mod (X^N - 1)$$

for some $a \in \mathbb{Z}_M$. We will say sometimes that F *passes through* a, or *is rooted at* a.

A set $A \subset \mathbb{Z}_M$ is *N-fibered in the p_ν direction* if it can be written as a union of disjoint N-fibers in the p_ν direction, all with the same multiplicity. In particular, A is *M-fibered in the p_ν direction* if there is a subset $A' \subset A$ such that $A = A' * F_\nu$.

If $N = p_1^{\alpha_1} \dots p_K^{\alpha_K}$ is a divisor of M, with $0 \le \alpha_\nu \le n_\nu$, we let

$$D(N) := p_1^{\gamma_1} \dots p_K^{\gamma_K}, \quad \text{where } \gamma_\nu = \max(0, \alpha_\nu - 1) \text{ for } \nu \in \{1, \dots, K\}.$$

We will also write $N_\nu = M/p_\nu$ for $\nu \in \{1, \dots, K\}$.

2.3. **Divisor set and box notation.** For $N|M$ and $A \subset \mathbb{Z}_M$, we define

$$\text{Div}_N(A) := \{(a - a', N) : \ a, a' \in A\} \tag{2.3}$$

When $N = M$, we will omit the subscript and write $\text{Div}(A) = \text{Div}_M(A)$. Informally, we will refer to the elements of $\text{Div}(A)$ as the *divisors of A* or *differences in A*. In cases when we need to indicate where a particular divisor of A must occur, we will use the following notation for localized divisor sets. If $A_1, A_2 \subset \mathbb{Z}_M$, we will write

$$\text{Div}_N(A_1, A_2) := \{(a_1 - a_2, N) : \ a_1 \in A_1, a_2 \in A_2\}. \tag{2.4}$$

If one of the sets is a singleton, say $A_1 = \{a_1\}$, we will simplify the notation and write $\text{Div}_N(a_1, A_2) = \text{Div}_N(\{a_1\}, A_2)$.

We will also use the N-box notation of [24], [25]. For $x \in \mathbb{Z}_M$, define

$$\mathbb{A}_m^N[x] = \#\{a \in A : \ (x - a, N) = m\}.$$

If $N = M$, we will usually omit the superscript and write $\mathbb{A}_m^M[x] = \mathbb{A}_m[x]$. For $X \subset \mathbb{Z}_M$ and $x \in \mathbb{Z}_M$, we define $\mathbb{A}_m^N[X] := \sum_{x' \in X} \mathbb{A}_m^N[x']$ and

$$\mathbb{A}_m^N[x|X] = \#\{a \in A \cap X : \ (x - a, N) = m\}.$$

2.4. **Standard tiling complements.** Suppose that $A\oplus B=\mathbb{Z}_M$. For $\nu\in\{1,\dots,K\}$, let

$$\mathfrak{A}_\nu(A)=\left\{\alpha_\nu\in\{1,2,\dots,n_\nu\}:\ \Phi_{p_\nu^{\alpha_\nu}}(X)|A(X)\right\}$$

The *standard tiling complement* $A^\flat\subset\mathbb{Z}_M$ is defined by

$$A^\flat(X)=\prod_{\nu=1}^{K}\prod_{\alpha_\nu\in\mathfrak{A}_\nu(A)}\left(1+X^{M_\nu p_\nu^{\alpha_\nu-1}}+\cdots+X^{(p_\nu-1)M_\nu p_\nu^{\alpha_\nu-1}}\right). \tag{2.5}$$

The set $A^\flat(X)$ has the same prime power cyclotomic divisors as $A(X)$, and satisfies (T2). If $s|M$ is a prime power, then $\Phi_s|A$ if and only if $\Phi_s\nmid B$ (see [2]), hence $A^\flat$ is uniquely determined by M and B. The following result was proved in [2], [24].

Proposition 2.1. [24, Proposition 3.4] *Let $A\oplus B=\mathbb{Z}_M$. Then $A^\flat\oplus B=\mathbb{Z}_M$ if and only if B satisfies (T2).*

We say that the tilings $A\oplus B=\mathbb{Z}_M$ and $A'\oplus B=\mathbb{Z}_M$ are *T2-equivalent* if

$$A\text{ satisfies (T2) }\Leftrightarrow A'\text{ satisfies (T2)}.$$

Since A and A' tile the same group $\mathbb{Z}_M$ with the same tiling complement B, they must have the same cardinality and the same prime power cyclotomic divisors. We will sometimes say simply that A is T2-equivalent to A' if both M and B are clear from context. Usually, A' will derived from A using certain permitted manipulations such as fiber shifts (Lemma 3.11). In particular, if we can prove that either A or B in a given tiling is T2-equivalent to a standard tiling complement, this resolves the problem completely in that case.

Corollary 2.2 ([24] Corollary 3.6)**.** *Suppose that the tiling $A\oplus B=\mathbb{Z}_M$ is T2-equivalent to the tiling $A^\flat\oplus B=\mathbb{Z}_M$. Then A and B satisfy (T2).*

2.5. **Cuboids.** Cuboids are an important tool in the literature on cyclotomic divisibility and Fuglede's conjecture, see e.g., [16], [17], [24], [25], [43]. We follow the presentation in [24, Section 5].

Definition 2.3. *(i) A* cuboid type *$\mathcal{T}$ on $\mathbb{Z}_N$ is an ordered triple $\mathcal{T}=(N,\vec{\delta},T)$, where:*

- *$N=\prod_{\nu=1}^K p_\nu^{n_\nu-\alpha_\nu}$ is a divisor of M, with $0\le\alpha_\nu\le n_\nu$ for each ν,*
- *$\vec{\delta}=(\delta_1,\dots,\delta_K)$, with $0\le\delta_\nu\le n_\nu-\alpha_\nu$,*
- *the* template *T is a nonempty subset of $\mathbb{Z}_N$.*

(ii) A cuboid *Δ of type $\mathcal{T}$ is a weighted multiset corresponding to a mask polynomial of the form*

$$\Delta(X)=X^c\prod_{\nu\in\mathfrak{J}}(1-X^{d_\nu}),$$

where $\mathfrak{J}=\mathfrak{J}_{\vec{\delta}}:=\{\nu:\delta_\nu\neq 0\}$, and c,d_ν are elements of $\mathbb{Z}_M$ such that $(d_\nu,N)=N/p_\nu^{\delta_\nu}$ for $\nu\in\{i,j,k\}$. The vertices *of Δ are the points*

$$x_{\vec{\epsilon}}=c+\sum_{\nu\in\mathfrak{J}}\epsilon_\nu d_\nu:\ \vec{\epsilon}=(\epsilon_\nu)_{\nu\in\mathfrak{J}}\in\{0,1\}^{|\mathfrak{J}|},$$

with weights $w_\Delta(x_{\vec{\epsilon}})=(-1)^{\sum_{\nu\in\mathfrak{J}}\epsilon_\nu}$.

(iii) Let $A \in \mathcal{M}(\mathbb{Z}_N)$, and let Δ be a cuboid of type $\mathcal{T}$. Define

$$\mathbb{A}^{\mathcal{T}}[\Delta] = \mathbb{A}^N_N[\Delta * T] = \sum_{\vec{\epsilon}\in\{0,1\}^k} w_\Delta(x_{\vec{\epsilon}})\mathbb{A}^N_N[x_{\vec{\epsilon}} * T],$$

where we recall that $x * T = \{x + t : \ t \in T\}$, so that

$$\mathbb{A}^N_N[x_{\vec{\epsilon}} * T] := \sum_{t\in T} \mathbb{A}^N_N[x_{\vec{\epsilon}} + t].$$

For consistency, we will also write $\mathbb{A}^{\mathcal{T}}[x] = \mathbb{A}^N_N[x * T]$ for $x \in \mathbb{Z}_M$.

An important special case is as follows: for $N|M$, an N-*cuboid* is a cuboid of type $\mathcal{T} = (N, \vec{\delta}, T)$, where $N|M$, $T(X) = 1$, and $\delta_\nu = 1$ for all ν such that $p_\nu|N$. Thus, N-cuboids have the form

$$\Delta(X) = X^c \prod_{p_\nu|N}(1 - X^{d_\nu}), \tag{2.6}$$

with $(d_\nu, N) = N/p_\nu$ for all ν such that $p_\nu|N$. We reserve the term "N-cuboid", without cuboid type explicitly indicated, to refer to cuboids as in (2.6); for cuboids of any other type, we will always specify $\mathcal{T}$.

Cuboids provide useful criteria to determine cyclotomic divisibility properties of mask polynomials. We say that a multiset $A \in \mathcal{M}(\mathbb{Z}_M)$ *is $\mathcal{T}$-null* if for every cuboid Δ of type $\mathcal{T}$,

$$\mathbb{A}^{\mathcal{T}}[\Delta] = 0.$$

For $A \in \mathcal{M}(\mathbb{Z}_N)$, we have $\Phi_N(X)|A(X)$ if and only if $\mathbb{A}^N_N[\Delta] = 0$ for every N-cuboid Δ. This has been known and used previously in the literature, see e.g. [43, Section 3], or [16, Section 3]. In particular, for any $N|M$, Φ_N divides A if and only if it divides the mask polynomial of $A \cap \Lambda(x, D(N))$ for every $x \in \mathbb{Z}_M$.

More generally, if for every $m|N$ the polynomial $\Phi_m(X)$ divides at least one of $A(X)$, $T(X)$, or $\Delta(X)$ for every Δ of type $\mathcal{T}(N, \vec{\delta}, T)$, then A is $\mathcal{T}$-null. We use such cuboid types to test for divisibility by combinations of cyclotomic polynomials. For example, we have the following.

- Let $n_i \geq 2$, and let $\mathcal{T} = (M, \vec{\delta}, 1)$, with $\delta_i = 2$ and $\delta_\nu = 1$ for all $\nu \neq i$. Then
$$\Phi_M\Phi_{M/p_i}|A \Leftrightarrow A \text{ is } \mathcal{T}\text{-null}.$$
- Assume that $n_i = 2$, and let $\mathcal{T} = (M, \vec{\delta}, T)$, where $\delta_i = 0$, $\delta_j = \delta_k = 1$, and
$$T(X) = \frac{X^{M/p_i} - 1}{X^{M/p_i^2} - 1} = 1 + X^{M/p_i^2} + \cdots + X^{(p_i-1)M/p_i^2}.$$
If $\Phi_M\Phi_{M/p_i^2}|A$, then A is $\mathcal{T}$-null.

3. Divisor isometries and the box product

Throughout this section, we assume that $M = p_1^{n_1} \dots p_K^{n_K}$, where $p_1, \dots, p_K$ are distinct primes. We do not need to assume that $K = 3$. Our analysis here is motivated in part by the fundamental theorems of Sands [38] and Tijdeman [48], which we now state.

Theorem 3.1. (Divisor exclusion; Sands [38]**)** *Let $A, B \subset \mathbb{Z}_M$. Then $A \oplus B = \mathbb{Z}_M$ if and only if $|A|\,|B| = M$ and*

$$\mathrm{Div}(A) \cap \mathrm{Div}(B) = \{M\}.$$

Theorem 3.2. (Dilation invariance of tiling; Tijdeman [48]**)** *Let $A \oplus B = \mathbb{Z}_M$, and let $r \in \mathbb{Z}_M$ satisfy $(r, |A|) = 1$. Then $rA \oplus B = \mathbb{Z}_M$ is also a tiling, where $rA = \{ra : a \in A\}$.*

3.1. Divisor isometries.

Definition 3.3. *Let $\psi : \mathbb{Z}_M \to \mathbb{Z}_M$ be a mapping. We will say that ψ is a* divisor isometry *if for every $x, x' \in \mathbb{Z}_M$ we have*

$$(\psi(x) - \psi(x'), M) = (x - x', M).$$

Any divisor isometry must be one-to-one. Indeed, if ψ is a divisor isometry and $\psi(x) = \psi(x')$ for some $x, x' \in \mathbb{Z}_M$, then $(x - x', M) = (\psi(x) - \psi(x'), M) = M$, so that $x = x'$. It is also easy to see that divisor isometries form a group: the composition of divisor isometries is also a divisor isometry, and if ψ is a divisor isometry, then so is ψ^{-1}.

The following lemma provides the rationale for the above definition.

Lemma 3.4. *Let $\psi : \mathbb{Z}_M \to \mathbb{Z}_M$ be a divisor isometry. If $A \oplus B = \mathbb{Z}_M$ is a tiling, then so is $\psi(A) \oplus B = \mathbb{Z}_M$.*

Proof. Since $\mathrm{Div}(\psi(A)) = \mathrm{Div}(A)$, this follows immediately from Theorem 3.1. □

We list several important examples.

Lemma 3.5. *The following mappings are divisor isometries.*

- Translations $\tau_c : \ x \to x - c$ *for any $c \in \mathbb{Z}_M$.*
- Invertible dilations $\psi_r : \ x \to rx$ *for any $r \in R := \{r \in \mathbb{Z}_M : \ (r, M) = 1\}$.*
- Plane exchange mappings*, defined as follows. Let $c, c' \in \mathbb{Z}_M$ with $c - c' = M/p_i^{\alpha_i+1}$ for some $i \in \{1, \dots, K\}$ and $0 \le \alpha_i < n_i$. Let $\mathsf{Ex}(c, c', p_i^{\alpha_i})$ be the mapping that "exchanges" the planes $\Pi(c, p_i^{\alpha_i})$ and $\Pi(c', p_i^{\alpha_i})$, so that*

$$\mathsf{Ex}(c, c', p_i^{\alpha_i})(x) = \begin{cases} x + (c' - c) & \text{if } x \in \Pi(c, p_i^{n_i - \alpha_i}), \\ x + (c - c') & \text{if } x \in \Pi(c', p_i^{n_i - \alpha_i}), \\ x & \text{if } x \notin \Pi(c, p_i^{n_i - \alpha_i}) \cup \Pi(c', p_i^{n_i - \alpha_i}). \end{cases}$$

Then $\mathsf{Ex}(c, c', p_i^{\alpha_i})$ is a divisor isometry.

Proof. The lemma follows immediately from the definition of a divisor isometry. For example, if $r \in R$, we have

$$(\psi_r(x) - \psi_r(x'), M) = (rx - rx', M) = (r(x - x'), M) = (x - x', M)$$

for all $x, x' \in \mathbb{Z}_M$. □

Lemma 3.6. (Properties of dilations) *For $r \in R$, define ψ_r as in Lemma 3.5.*

(i) *If $x \in \mathbb{Z}_M$ and $D|M$, then $\psi_r(\Lambda(x, D)) = \Lambda(rx, D)$. (In particular, $\psi_r(\Pi(x, p_\nu^\alpha)) = \Pi(rx, p_\nu^\alpha)$.)*

(ii) *Let* $x, x' \in \mathbb{Z}_M$ *with* $(x, M) = (x', M) = m$*. Let*

$$R_{x,x'} := \{r \in R : \ \psi_r(x) = x'\}.$$

Then

$$|R_{x,x'}| = \frac{\phi(M)}{\phi(M/m)}.$$

Moreover, for any $r \in R_{x,x'}$ *we have*

$$R_{x,x'} = \Lambda(r, M/(x, M)) \cap R.$$

Proof. To prove (i), we note that $x, x' \in \mathbb{Z}_M$ satisfy $D|(x - x')$, if and only if $D|(rx - rx')$ for $r \in R$. This implies the claim.

We now prove (ii). Let $x, x' \in \mathbb{Z}_M$ with $(x, M) = (x', M) = m$, so that $x/m, x'/m$ are well defined elements of $\mathbb{Z}_M$. We first check that there exists at least one $r \in R$ such that $rx = x'$. Indeed, since $(x/m, M) = 1$, the element x/m is invertible in $\mathbb{Z}_M$. Let $u \in \mathbb{Z}_M$ be its inverse, so that $ux/m = 1$. Clearly, $(u, M) = 1$. Let $r = ux'/m$, then $(r, M) = 1$ and $rx = (ux'/m)x = x'(ux/m) = x'$.

With r as above, we have $r' \in R_{x,x'}$ if and only if $r \in R$ and $rx = r'x = x'$, so that $(r - r')x = 0$ in $\mathbb{Z}_M$. Since $(x, M) = m$, we need $r - r'$ to be divisible by M/m. In other words, $r' \in R \cap \Lambda(r, M/m)$.

We now count the number Y of elements $r' \in \Lambda(r, M/m)$ such that $(r', M) = 1$. Without loss of generality, we may assume that $M/m = p_1^{\alpha_1} \dots p_l^{\alpha_l}$ for some $l \leq K$. Then $|\Lambda(r, M/m)| = m$, and

$$\begin{aligned} Y &= m \cdot \frac{p_{l+1} - 1}{p_{l+1}} \dots \frac{p_K - 1}{p_K} \\ &= p_1^{n_1 - \alpha_1} \dots p_l^{n_l - \alpha_l} p_{l+1}^{n_{l+1} - 1} \dots p_K^{n_K - 1} (p_{l+1} - 1) \dots (p_K - 1), \end{aligned}$$

so that $Y \phi(M/m) = \phi(M)$ as claimed. □

Lemma 3.7. (Transitivity of dilations) *For* $\nu = 1, \dots, K$*, let* $x_\nu, x'_\nu \in \mathbb{Z}_M$ *with* $(x_\nu, M) = (x'_\nu, M) = M/p_\nu^{\alpha_\nu}$*, where* $\alpha_1, \dots, \alpha_K \geq 1$*. Then there exists* $r \in R$ *such that* $rx_\nu = x'_\nu$ *for* $\nu = 1, \dots, K$*.*

Proof. By Lemma 3.6, for each ν there exists $r_\nu \in R$ such that $r_\nu x_\nu = x'_\nu$. Furthermore,

$$R_{x_\nu, x'_\nu} = \Pi(r_\nu, p_\nu^{\alpha_\nu}) \cap R.$$

Let $r \in \bigcap_{\nu=1}^K \Pi(r_\nu, p_\nu^{\alpha_\nu})$. We claim that $r \in R$. Indeed, suppose that $p_\nu | r$ for some ν. Since $p_\nu^{\alpha_\nu} | r - r_\nu$ and $\alpha_\nu \geq 1$, this implies that $p_\nu | r_\nu$, contradicting the fact that $r_\nu \in R$. This establishes the claim, and proves that r satisfies the conclusion of the lemma. □

A brief discussion of the relationship between Lemma 3.4, Theorem 3.2, and Theorem 3.1 is in order. The special case of Tijdeman's theorem with $r \in R$ was first proved by Sands [38, Theorem 2]. Sands used this to prove Theorem 3.1 [38, Theorem 3]. Since the dilation ψ_r with $r \in R$ is a divisor isometry, Lemma 3.4 includes such dilations as a special case. The simplest way to prove the lemma from scratch is to start with ψ_r for $r \in R$ (as in [38, Theorem 2]), deduce Theorem 3.1, and then use it to extend the conclusion to more general divisor isometries. (We note that a composition of any number of translations and invertible dilations is always a linear mapping of the form $\psi(x) = rx + c$ with $r \in R$. There are divisor isometries, for example most plane exchange mappings, that do not have this form.)

Theorem 3.2 is more general. Instead of assuming as in [38, Theorem 2] that $r \in R$, Tijdeman only assumes that r is relatively prime to $|A|$. This is a significantly weaker assumption than that in [38, Theorem 2], since the mapping $x \to rx$ with $(r, |A|) = 1$ need not be a divisor isometry if $|B|$ has prime factors that $|A|$ does not have. Coven and Meyerowitz [2] used the full strength of Theorem 3.2 that if $|A|$ tiles the integers, then it must in fact tile a cyclic group $\mathbb{Z}_M$, where M has the same prime factors as $|A|$.

An alternative proof of Theorem 3.1, based on Theorem 3.8 below, is given in [24] and [11]. While the proof of Theorem 3.8 in [11], [24] is harmonic-analytic, the same identity can also be proved using Tijdeman's theorem (in fact, [38, Theorem 2] is also sufficient for this purpose). We do this below in Section 3.2.

3.2. **A combinatorial interpretation of the box product.** Following [24], we define the *M-box product* as follows. If $A, B \subset \mathbb{Z}_M$, let

$$\langle \mathbb{A}[x], \mathbb{B}[y] \rangle = \sum_{m|M} \frac{1}{\phi(M/m)} \mathbb{A}_m[x]\mathbb{B}_m[y].$$

Here ϕ is the Euler totient function: if $n = \prod_{\iota=1}^{L} q_\iota^{r_\iota}$, where $q_1, \dots, q_L$ are distinct primes and $r_\iota \in \mathbb{N}$, then $\phi(n) = \prod_{\iota=1}^{L}(q_\iota - 1)q_\iota^{r_\iota - 1}$.

Theorem 3.8. *(*[24]*; following* [11, Theorem 1]*) If $A \oplus B = \mathbb{Z}_M$ is a tiling, then*

$$\langle \mathbb{A}^M[x], \mathbb{B}^M[y] \rangle = 1 \quad \forall x, y \in \mathbb{Z}_M. \tag{3.1}$$

The proof of Theorem 3.8 in [11], [24] uses discrete harmonic analysis. Below, we present an alternative combinatorial proof, based on averaging over dilations ψ_r with $r \in R$.

Proof. Suppose that $A \oplus B = \mathbb{Z}_M$. Without loss of generality, we may assume that $x = y = 0$. By Theorem 3.2, we have $rA \oplus B = \mathbb{Z}_M$ for all $r \in R$.

We count the number of triples $(a, b, r) \in A \times B \times R$ such that $ra + b = 0$. We have $|R| = \phi(M)$, and for every $r \in R$, it follows from Theorem 3.2 that there is exactly one pair $(a, b) \in A \times B$ such that $ra + b = 0$. Thus the number of such triples is $\phi(M)$.

On the other hand, let $m|M$, and suppose that $b \in B$ satisfies $(b, M) = m$. Then $(-b, M) = m$, and by Lemma 3.6, for every $a \in A$ with $(a, M) = m$ we have

$$|\{r \in R : \ \psi_r(a) = -b\}| = \frac{\phi(M)}{\phi(M/m)}.$$

Thus the number of triples as above is also equal to

$$\sum_{m|M} \frac{\phi(M)}{\phi(M/m)} \mathbb{A}_m[0]\mathbb{B}_m[0].$$

The identity follows. $\square$

3.3. **Saturating sets.** Let $A \oplus B = \mathbb{Z}_M$, and $x, y \in \mathbb{Z}_M$. We define the sets $A_{x,y}$ and $B_{y,x}$ to be the sets that saturate the box product on the left side of (3.1):

$$A_{x,y} := \{a \in A : \ (x-a, M) = (y-b, M) \text{ for some } b \in B\},$$

and similarly for $B_{y,x}$ with A and B interchanged, so that

$$\langle \mathbb{A}[x], \mathbb{B}[y] \rangle = \sum_{m|M} \frac{1}{\phi(M/m)} \mathbb{A}_m[x|A_{x,y}] \mathbb{B}_m[y|B_{y,x}].$$

The *saturating set* for x is

$$A_x := \{a \in A : (x-a, M) \in \text{Div}(B)\} = \bigcup_{b \in B} A_{x,b},$$

with B_y defined similarly.

We note an alternative description of saturating sets, based on dilations and closely related to the alternative proof of Theorem 3.8 presented above.

Lemma 3.9. *Let $A \oplus B = \mathbb{Z}_M$ be a tiling, and let $x, y \in \mathbb{Z}_M$, $a \in A$, $b \in B$. Then the following are equivalent:*

(i) *$a \in A_{x,y}$ and $b \in B_{y,x}$,*
(ii) *there exists $r \in R$ such that $x - a = r(y - b)$,*
(iii) *there exists $r \in R$ such that $0 = (a-x)+(-r)(b-y)$ in the tiling $\tau_x(A) \oplus (-r)\tau_y(B) = \mathbb{Z}_M$, where we use τ_x to denote translations as in Lemma 3.5.*

Proof. The equivalence between (i) and (ii) follows from Lemma 3.6 (ii). Part (iii) is a reformulation of (ii). □

By Theorem 3.1, $A_a = \{a\}$ for all $a \in A$. For $x \in \mathbb{Z}_M \setminus A$, A_x must be nonempty by (3.1), and obeys the following geometric constraints. For $x, x' \in \mathbb{Z}_M$ such that $(x - x', M) = p_1^{\alpha_1} \dots p_K^{\alpha_K}$, where $0 \le \alpha_\nu \le n_\nu$, define

$$\text{Span}(x, x') = \bigcup_{\nu : \alpha_\nu < n_\nu} \Pi(x, p_\nu^{\alpha_\nu + 1}),$$

$$\text{Bispan}(x, x') = \text{Span}(x, x') \cup \text{Span}(x', x).$$

Then for any $x, x', y \in \mathbb{Z}_M$, we have

$$A_{x',y} \subset A_{x,y} \cup \text{Bispan}(x, x'), \tag{3.2}$$

and in particular,

$$A_x \subset \bigcap_{a \in A} \text{Bispan}(x, a). \tag{3.3}$$

In an important special case, if $x \in \mathbb{Z}_M \setminus A$ satisfies $(x - a, M) = M/p_i$ for some $a \in A$, then

$$A_x \subset \text{Bispan}(x, a) = \Pi(x, p_i^{n_i}) \cup \Pi(a, p_i^{n_i}).$$

Our evaluations of saturating sets will always begin with geometric restrictions based on (3.3).

3.4. **Cofibered structures.** The following is a simplified version of the definitions and results of [24, Section 8], restricted to $M = p_i^2 p_j^2 p_k^2$. Most of this article can be read with just these definitions, if the reader is willing to substitute $n_i = n_j = n_k = 2$ in all arguments. On those few occasions when the more general versions are necessary even in that case, we have to refer the reader to [24, Section 8].

If $F \subset \mathbb{Z}_M$ is an M-fiber in the p_ν direction, we say that an element $x \in \mathbb{Z}_M$ is at *distance* m from F if $m|M$ is the maximal divisor such that $(z - x, M) = m$ for some $z \in F$. It is easy to see that such m exists.

Let $A \oplus B = \mathbb{Z}_M$ be a tiling. We will often be interested in finding "complementary" fibers and fibered structures in A and B, in the following sense.

Definition 3.10 (Cofibers and cofibered structures). *Let $A, B \subset \mathbb{Z}_M$, with $M = p_i^2 p_j^2 p_k^2$, and let $\nu \in \{i, j, k\}$.*

(i) We say that $F \subset A, G \subset B$ are $(1,2)$-cofibers *in the p_ν direction if F is an M-fiber and G is an M/p_ν-fiber, both in the p_ν direction.*

(ii) We say that the pair (A, B) has a (1,2)-cofibered structure *in the p_ν direction if*

- *B is M/p_ν-fibered in the p_ν direction,*
- *A contains at least one "complementary" M-fiber $F \subset A$ in the p_ν direction, which we will call a* cofiber *for this structure.*

The advantage of cofibered structure is that it permits fiber shifts as described below. In many cases, we will be able to use this to reduce the given tiling to a simpler one.

Lemma 3.11 (Fiber-Shifting Lemma). *Let $A \oplus B = \mathbb{Z}_M$. Assume that the pair (A, B) has a $(1,2)$-cofibered structure, with a cofiber $F \subset A$. Let A' be the set obtained from A by shifting F to a point $x \in \mathbb{Z}_M$ at a distance M/p_i^2 from it. Then $A' \oplus B = \mathbb{Z}_M$, and A is T2-equivalent to A'.*

In order to find cofibered structures in (A, B), we will typically use saturating sets, via the following lemma.

Lemma 3.12. *Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling, with $M = p_i^2 p_j^2 p_k^2$. Suppose that $x \in \mathbb{Z}_M \setminus A$, $b \in B$, $M/p_\nu \in \mathrm{Div}(A)$, and $A_{x,b} \subset \ell_\nu(x)$ for some $\nu \in \{i, j, k\}$ and $b \in B$. Then*

$$\mathbb{A}^M_{M/p_\nu^2}[x] \mathbb{B}^M_{M/p_\nu^2}[b] = \phi(p_\nu^2). \tag{3.4}$$

with the product saturated by a $(1,2)$-cofiber pair (F, G) such that $F \subset A$ is at distance M/p_i^2 from x and $G \subset B$ is rooted at b. In particular, if $A_x \subset \ell_\nu(x)$, then the pair (A, B) has a $(1,2)$-cofibered structure.

4. Splitting

Definition 4.1. *Let $M = p_1^{n_1} \dots p_K^{n_K}$, and assume that $A \oplus B = \mathbb{Z}_M$ is a tiling. For a set $Z \subset \mathbb{Z}_M$, define*

$$\Sigma_A(Z) = \{a \in A : \ z = a + b \text{ for some } z \in Z, \ b \in B\},$$
$$\Sigma_B(Z) = \{b \in B : \ z = a + b \text{ for some } z \in Z, \ a \in A\}.$$

These are the sets of elements of A and B that tile Z. Note that $\Sigma_A(Z)$ depends on both A and B. This is important when more than one tiling complement of A is being considered, for example $A \oplus B = \mathbb{Z}_M$ and $A \oplus \psi(B) = \mathbb{Z}_M$ for a divisor isometry ψ. In such situation, we will identify the relevant tiling explicitly. If $Z = \{z\}$ is a singleton, we will simplify the notation and write $\Sigma_A(z) = \Sigma_A(\{z\})$.

It will also be convenient to use the following terminology.

Definition 4.2. *For $u, v \in \mathbb{Z}_M$, we will say that u and v* match in the p_i direction *if $(u, p_i^{n_i}) = (v, p_i^{n_i})$, and that they* do not match *or* are mismatched *in the p_i direction if $(u, p_i^{n_i}) \neq (v, p_i^{n_i})$*

The intended application is when $u = a - a'$ and $v = b - b'$ for some $a, a' \in A$ and $b, b' \in B$. By Theorem 3.1, if $A \oplus B = \mathbb{Z}_M$, it is not possible for $a - a'$ and $b - b'$ to match in all directions unless $a = a'$ and $b = b'$. On the other hand, suppose that $a + b = z$ and $a' + b' = z'$. Then the only directions in which $a - a'$ and $b - b'$ are permitted to be mismatched are those of the prime factors of $M/(z - z', M)$ (see Lemma 4.3 below). This will lead to geometric constraints on $\Sigma_A(Z)$ and $\Sigma_B(Z)$ for various grids Z, analogous to those provided by Lemma 3.3 for saturating sets.

Lemma 4.3. *Let $M = p_1^{n_1} \dots p_K^{n_K}$. Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling. Let $z, z' \in \mathbb{Z}_M$, and let $a, a' \in A, b, b' \in B$ satisfy $a + b = z$ and $a' + b' = z'$. Then:*

- (i) *For any $m | z - z'$, we have $(a - a', m) = (b - b', m)$.*
- (ii) *For every i such that $p_i^{n_i} \mid (z - z')$, $a - a'$ and $b - b'$ must match in the p_i direction.*
- (iii) *Suppose that $a - a'$ and $b - b'$ do not match in the direction p_j, and let β be the exponent such that $p_j^\beta \parallel z - z'$. Then $p_j^\beta | a - a'$ and $p_j^\beta | b - b'$.*

Proof. We have

$$(a - a') + (b - b') = (a + b) - (a' + b') = z - z',$$

which implies (i).

For (ii), let $m = (z - z', M)$, then $(a - a', m) = (b - b', m)$ by the first part of the lemma. If $p_i^{n_i} \mid (z - z')$, then $p_i^{n_i} | m$, so that $(a - a', p_i^{n_i}) = (b - b', p_i^{n_i})$ as claimed.

Finally, to prove (iii), suppose for contradiction that $p_j^\gamma \parallel a - a'$ for some $0 \leq \gamma < \beta$. Then we also have $p_j^\gamma \parallel b - b'$, contradicting the assumption that p_j is a direction of mismatch. □

Of particular interest is the case where only one direction of mismatch is permitted.

Definition 4.4. *Let $M = p_1^{n_1} \dots p_K^{n_K}$, and let $N = M/p_i^\alpha$ for some $i \in \{1, \dots, K\}$ and $0 \leq \alpha < n_i$. Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling. Let $Z \subset \mathbb{Z}_M$ be an N-fiber in the p_i direction, with $|Z| \geq 2$. We will say that Z* splits with parity (A, B) *in the p_i direction on the scale N if:*

- (i) *$p_i^{n_i - \alpha} | a - a'$ for any $a, a' \in \Sigma_A(Z)$ (in other words, $\Sigma_A(Z) \subset \Pi(a, p_i^{n_i - \alpha})$ for any $a \in \Sigma_A(Z)$),*
- (ii) *$p_i^{n_i - \alpha - 1} \parallel b - b'$ for any two distinct $b, b' \in \Sigma_B(Z)$. In particular, $\Sigma_B(Z) \subset \Pi(b, p_i^{n_i - \alpha - 1})$ for any $b \in \Sigma_B(Z)$.*

Since Z in the above definition is assumed to be an N-fiber in the p_i direction, both the scale N and the direction of splitting can be identified from the choice of Z. Therefore we will often just say that Z *splits* with parity (A,B), or simply that it *splits* if the parity is unknown.

In our proof of (T2), we only use splitting on the scale M, so that $\alpha=0$ and $N=M$. We recommend making this substitution on the first reading, since this case already captures all of the main ideas here. However, the extension to lower scales is likely to be useful in future work and requires only minimal additional effort, hence we include it in the results below.

Lemma 4.5. (Splitting for fibers) *Let $M=p_1^{n_1}\dots p_K^{n_K}$, and let $N=M/p_i^\alpha$ for some $i\in\{1,\dots,K\}$ and $0\le\alpha<n_i$. Assume that $A\oplus B=\mathbb{Z}_M$ is a tiling. Then any N-fiber in the p_i direction splits in that direction.*

More explicitly, we have the following. Let $z_0,z_1,\dots,z_{p_i-1}\in\mathbb{Z}_M$ satisfy $(z_\nu-z_{\nu'},M)=M/p_i^{n_i-\alpha-1}$ for $\nu\neq\nu'$. Let $a_\nu\in A, b_\nu\in B$ be such that $a_\nu+b_\nu=z_\nu$. Then one of the following must hold:

(i) (Splitting with parity (A,B)) *We have $a_1,\dots,a_{p_i-1}\in\Pi(a_0,p_i^{n_i-\alpha})$ and $p_i^{n_i-\alpha-1}\parallel b_\nu-b_{\nu'}$ for $\nu\neq\nu'$,*

(ii) (Splitting with parity (B,A)) *We have $b_1,\dots,b_{p_i-1}\in\Pi(b_0,p_i^{n_i-\alpha})$ and $p_i^{n_i-\alpha-1}\parallel a_\nu-a_{\nu'}$ for $\nu\neq\nu'$.*

Proof. Without loss of generality, we may assume that $z_0=0$. Let $\nu\neq\nu'$. By Lemma 4.3 (ii), $a_\nu-a_{\nu'}$ and $b_\nu-b_{\nu'}$ match in all directions except p_i. By divisor exclusion, they must be mismatched in the p_i direction. Applying Lemma 4.3 again, this time with $m=p_i^{n_i-\alpha-1}$, we get

$$(a_\nu-a_{\nu'},p_i^{n_i-\alpha-1})=(b_\nu-b_{\nu'},p_i^{n_i-\alpha-1}).$$

By Lemma 4.3 (iii), we must have $a_\nu\in\Pi(a_0,p_i^{n_i-\alpha-1})$ and $b_\nu\in\Pi(b_0,p_i^{n_i-\alpha-1})$ for all ν. Additionally, since $p_i^{n_i-\alpha}$ does not divide $z_\nu-z_{\nu'}$, it does not divide at least one of $a_\nu-a_{\nu'}$ and $b_\nu-b_{\nu'}$. The only way to reconcile this with the mismatch is to have either

$$p_i^{n_i-\alpha}|a_\nu-a_{\nu'} \text{ and } (b_\nu-b_{\nu'},M)=(a_\nu-a_{\nu'},M/p_i^{n_i-\alpha-1}), \tag{4.1}$$

or the same with A and B interchanged.

Assume now that (4.1) holds for some ν,ν'. Then for any other $\nu''\in\{0,1,\dots,p_i-1\}$, we must have either $p_i^{n_i-\alpha}\nmid b_\nu-b_{\nu''}$ or $p_i^{n_i-\alpha}\nmid b_{\nu'}-b_{\nu''}$; in both cases, it follows that $p_i^{n_i-\alpha}|a_\nu-a_{\nu''}$, and we are in the first case of the lemma (splitting with (A,B) parity). □

The next lemma relates splitting to saturating sets.

Lemma 4.6. *Suppose that $0\in A\cap B$ and set $z_0=0$. The following are equivalent:*

(i) $(a-\nu M/p_i,M)=(b-\nu' M/p_i,M)$ *for some $\nu,\nu'\in\{0,\dots,p_i-1\}$ (in other words, $a\in A_{x,y}$ and $b\in B_{y,x}$ for $x=\nu M/p_i$ and $y=\nu' M/p_i$),*

(ii) $a+rb=\nu'' M/p_i$ *for some $r\in R$ and $\nu''\in\{0,\dots,p_i-1\}$.*

Proof. We have that (i) holds if and only if $a-\nu M/p_i=-r(b-\nu' M/p_i)$, which in turn is equivalent to $a+rb=(\nu-r\nu')M/p_i$ □

Lemma 4.7. (Plane consistency) *Let $M=p_1^{n_1}\dots p_K^{n_K}$. Assume that $A\oplus B=\mathbb{Z}_M$ is a tiling. Fix $i,j\in\{1,\dots,K\}$, and let $\Lambda:=\Lambda(z,M/p_i^{\alpha_i}p_j^{\alpha_j})$ for some $z\in\mathbb{Z}_M$, $1\le\alpha_i\le n_i-1$, and $1\le\alpha_j\le n_j-1$. Then there exists a $\nu\in\{i,j\}$ such that*

$$\Sigma_A(\Lambda)\subset A\cap\Pi(a,p_\nu^{n_\nu-\alpha_\nu}),$$

$$\Sigma_B(\Lambda) \subset B \cap \Pi(b, p_\nu^{n_\nu - \alpha_\nu}),$$

where $a \in A$ and $b \in B$ satisfy $a + b = z$.

Proof. Suppose for contradiction that the lemma is false. Then there exist $z', z'' \in \Lambda$ such that $z' = a' + b'$ and $z'' = a'' + b''$ with $a', a'' \in A$ and $b', b'' \in B$, and

$$p_i^{n_i-\alpha_i} \nmid a - a', \quad p_j^{n_j-\alpha_j} \nmid b - b''. \tag{4.2}$$

Let z_{ij} be the unique element of $\Pi(z', p_j^{n_j}) \cap \Pi(z'', p_i^{n_i}) \cap \Lambda$, so that

$$M/p_i^{\alpha_i} \mid z_{ij} - z', \quad M/p_j^{\alpha_j} \mid z_{ij} - z''.$$

By Lemma 4.5 with $N = M/p_i^{\alpha_i-1}p_j^{\alpha_j-1}$, we have $z_{ij} = a_{ij} + b_{ij}$ for some $a_{ij} \in A$ and $b_{ij} \in B$ such that

$$p_i^{n_i-\alpha_i} \mid a_{ij} - a', \quad p_j^{n_j-\alpha_j} \mid b_{ij} - b''.$$

Together with (4.2), this implies that

$$p_i^{n_i-\alpha_i} \nmid a - a_{ij} \text{ and } p_j^{n_j-\alpha_j} \nmid b - b_{ij}.$$

However, we also have $M/p_i^{\alpha_i}p_j^{\alpha_j} | z - z_{ij}$. It follows from Lemma 4.3 (iii) that $a - a_{ij}$ and $b - b_{ij}$ match in the p_i and p_j directions, and by Lemma 4.3 (ii), they also match in all other directions. This contradicts divisor exclusion and ends the proof of the lemma. □

5. Tiling reductions

5.1. **Tiling reductions for 3 primes.** The tiling reductions below allow us, under certain assumptions, to decompose a tiling $A \oplus B = \mathbb{Z}_M$ into a family of tilings of $\mathbb{Z}_{M/p_\nu}$ for some ν, with the additional property that if (T2) holds for both sets in each of the smaller tilings, then it also holds for A and B. The subgroup reduction is due to Coven and Meyerowitz [2]; the formulation we use is from [24, Theorem 6.1]. The slab reduction is from [24, Theorem 6.5 and Corollary 6.6]. Both reductions are valid for tilings of $\mathbb{Z}_M$ with no assumptions on the number of distinct prime factors.

In order to deduce (T2) for A and B, we must know that (T2) holds for both sets in the smaller tilings. If $M = p_i^2p_j^2p_k^2$, this is provided by [24, Corollary 6.2] (based on the methods of [2]). Theorem 5.1 and Corollary 5.3 combine the two steps, in a form ready to apply here with $M = p_i^2p_j^2p_k^2$

An additional goal of this section is to relate the slab reduction conditions in Theorem 5.2 to the concept of splitting. This relationship is made precise in Lemma 5.6. This part does not require any assumptions on the prime factorization of M, hence we will use the general formulation of Theorem 5.2 stated below.

A connection to a few of the conjectures posed in [24, Section 9] should also be mentioned. In particular, Theorem 5.6 below establishes a direct link between the slab reduction and [24, Conjecture 9.4], while Corollary 5.7 (i) may be viewed as a step towards resolving Conjecture 9.2 in [24].

Theorem 5.1. (Subgroup reduction) [2, Lemma 2.5] *Let $M = p_i^{n_i}p_j^{n_j}p_k^{n_k}$. Assume that $A \oplus B = \mathbb{Z}_M$, and that $A \subset p_\nu\mathbb{Z}_M$ for some $\nu \in \{i, j, k\}$ such that $p_\nu \parallel |B|$. Then A and B satisfy (T2).*

Theorem 5.2. *Let $M = p_1^{n_1} \dots p_K^{n_K}$. Assume that $A \oplus B = \mathbb{Z}_M$, and that $\Phi_{p_i^{n_i}} | A$ for some $i \in \{1, \dots, K\}$. Define*

$$A_{p_i} = \{a \in A : \ 0 \le \pi_i(a) \le p_i^{n_i - 1} - 1\}.$$

Then the following are equivalent:

(i) *For any translate A' of A, we have $A'_{p_i} \oplus B = \mathbb{Z}_{M/p_i}$.*
(ii) *For every m such that $p_i^{n_i} | m | M$, we have*

$$m \in \mathrm{Div}(A) \Rightarrow m/p_i \notin \mathrm{Div}(B). \tag{5.1}$$

(iii) *For every d such that $p_i^{n_i} | d | M$, at least one of the following holds:*

$$\Phi_d | A,$$

$$\Phi_{d/p_i} \Phi_{d/p_i^2} \dots \Phi_{d/p_i^{n_i}} \mid B.$$

Corollary 5.3. (Slab reduction) *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. Assume that $A \oplus B = \mathbb{Z}_M$, and that there exists a $\nu \in \{i, j, k\}$ such that $\Phi_{p_\nu^{n_\nu}} | A$, $p_\nu \parallel |A|$, and A, B obey the condition (ii) of Theorem 5.2 with $i = \nu$. (In particular, this holds if A is M-fibered in one of the p_i, p_j, p_k directions.) Then A and B satisfy (T2).*

The assumption that $p_\nu \parallel |B|$ in Theorem 5.1 guarantees that in any tiling $A' \oplus B' = \mathbb{Z}_{N_\nu}$ with $|A'| = |A|$ and $|B'| = |B|/p_\nu$, $|A'|$ and $|B'|$ have only two common factors. Hence the assumption (ii) of [24, Theorem 6.1] is satisfied by [24, Corollary 6.2], and we deduce that A and B both satisfy (T2). The assumption that $p_\nu \parallel |A|$ in Corollary 5.3 serves the same purpose. Without these assumptions, we can still use Theorem 5.1 and Corollary 5.3 to reduce proving (T2) for A and B to proving it for a family of tilings of $\mathbb{Z}_{M/p_\nu}$; however, the condition (T2) for those tilings is no longer guaranteed.

5.2. **Splitting and slab reduction.** This section aims to connect the slab reduction with uniform splitting parity in a fixed direction. Clearly, if (A, B) satisfy (5.1) then the same holds for (A, rB) for all $r \in R$. Thus (A, B) satisfy the slab reduction conditions if and only if (A, rB) satisfy the slab reduction conditions for any $r \in R$. Throughout this subsection we will set $M = p_1^{n_1} \dots p_K^{n_K}$.

Definition 5.4. *Let $A \oplus B = \mathbb{Z}_M$. We say that the tiling $A \oplus B = \mathbb{Z}_M$ has* uniform (A, B) splitting parity in the p_i direction *if all M-fibers in the p_i direction split with parity (A, B). Uniform (B, A) splitting parity is defined analogously.*

Definition 5.5. *Let $A \oplus B = \mathbb{Z}_M$. Given $a \in A$ and $b \in B$, we say that the product* $\langle \mathbb{A}[a * F_i], \mathbb{B}[b] \rangle$ splits with parity (A, B) *if $A_{x,b} \subset \Pi(a, p_i^{n_i})$ for all $x \in a * F_i$, and* with parity (B, A) *if $A_{x,b} \subset \Pi(x, p_i^{n_i})$ for all $x \in a * F_i$.*

Lemma 5.6. *Let $A \oplus B = \mathbb{Z}_M$. Suppose that $\Phi_{p_i^{n_i}} | A$. Then the following are equivalent:*

(I) *(A, B) satisfy the slab reduction conditions in Theorem 5.2.*
(II) *The tiling $A \oplus rB = \mathbb{Z}_M$ has uniform (rB, A) splitting parity in the p_i direction, for all $r \in R$.*
(III) *For every $a \in A$ and $b \in B$, the product $\langle \mathbb{A}[a * F_i], \mathbb{B}[b] \rangle$ splits with parity (B, A).*

Proof. We first prove that (I) and (II) are equivalent. Suppose that (A, B) satisfy the slab reduction conditions. Note that the condition (ii) of Theorem 5.2 depends only on the sets $\mathrm{Div}(A)$ and $\mathrm{Div}(B)$. Since $\mathrm{Div}(B) = \mathrm{Div}(rB)$ for $r \in R$, it follows that (A, rB) must also satisfy the slab reduction conditions for such r. Assume, by contradiction, that there exist $z \in \mathbb{Z}_M$ and $r \in R$ such that $z * F_i$ splits with parity (A, rB) in the tiling $A \oplus rB = \mathbb{Z}_M$. Let $a \in A, b \in B$ satisfy $z = a + rb$. Without loss of generality, we may assume that $a \in A_{p_i}$. We prove that (A_{p_i}, rB) do not tile $\mathbb{Z}_{M/p_i}$.

By the parity assumption, there exist

$$a_1, \dots, a_{p_i-1} \in A \cap \Pi(a, p_i^{n_i}) \subset A_{p_i}$$

and

$$b_1, \dots, b_{p_i-1} \in B \cap (\Pi(b, p_i^{n_i-1}) \setminus \Pi(b, p_i^{n_i}))$$

such that $a_\mu + rb_\mu = z + \mu M/p_i, \mu = 1, \dots, p_i - 1$. It follows that $(a + rb, M/p_i) = (a_\mu + rb_\mu, M/p_i)$ for all $\mu \in \{1, \dots, p_i - 1\}$, which is a contradiction, since all sums $a + rb$ with $a \in A_{p_i}$ and $b \in B$ must be distinct modulo M/p_i.

For the other direction, assume by contradiction that (II) holds but (A, B) do not satisfy the slab reduction conditions. Then (5.1) must fail, so that there exist $p_i^{n_i} | m | M$ such that $m \in \mathrm{Div}(A)$ and $m/p_i \in \mathrm{Div}(B)$. Without loss of generality, we may assume that

$$0 \in A \cap B \tag{5.2}$$

and there exist $a \in A, b \in B$ and $\mu \in \{1, \dots, p_i - 1\}$ such that

$$(a, M) = (b - \mu M/p_i, M) = m.$$

By Lemma 4.6, there exists $r \in R$ such that $r(b - \mu M/p_i) = -a$, so that $a + rb = r\mu M/p_i \in F_i$. By (5.2), the latter implies that F_i splits with parity (A, rB), contradicting (II).

Next, we prove that (II) and (III) are equivalent. We show that (II) implies (III), but the argument is completely reversible. Indeed, suppose that (II) holds yet (III) fails. Without loss of generality, we may assume that $0 \in A \cap B$ and that (III) fails with $a = b = 0$ and $x = M/p_i$, so that $A_{M/p_i,0} \cap \Pi(0, p_i^{n_i}) \neq \emptyset$. This implies that there exist $p_i^{n_i} | m | M$ and

$$a \in A \cap \Pi(0, p_i^{n_i}), b \in B \cap \Pi(M/p_i, p_i^{n_i}) \tag{5.3}$$

satisfying

$$(a, M) = (b - M/p_i, M) = m.$$

By Lemma 4.6, there exists $r \in R$ such that $r(b - M/p_i) = -a$, so that $a + rb = \mu M/p_i$ for some $\mu \in \{1, \dots p_i - 1\}$. By (5.3), this means that the tiling pair (A, rB) does not have uniform splitting in the p_i direction with parity (rB, A), contradicting (II). □

Corollary 5.7. *Let $A \oplus B = \mathbb{Z}_M$ be a tiling. Assume that $\Phi_{p_i^{n_i}} | A$ for some $i \in \{1, \dots, K\}$, and that at least one of the following holds:*

(i) $\mathbb{A}_{M/p_i}[a] > 0$ *for every* $a \in A$,
(ii) *for every* $b \in B$, *we have*

$$|B \cap \Pi(b, p_i^{n_i})| = |B|/(|B|, p_i^{n_i}). \tag{5.4}$$

Then A satisfies the conditions of Theorem 5.2.

Proof. Suppose first that (i) holds. This clearly implies the condition (II) of Lemma 5.6, hence the conclusion follows from the lemma.

Assume now that (ii) holds. Since $\Phi_{p_i^{n_i}}|A$, we must have

$$(|B|, p_i^{n_i}) = \prod_{\alpha:1\leq\alpha\leq n_i-1, \Phi_{p_i^\alpha}|B} \Phi_{p_i^\alpha}(1).$$

It follows as in [25, Lemma 4.3 and Corollary 4.4] (reproduced here as Lemma 7.3 and Corollary 7.4) that for every $b\in B$,

$$|B\cap\Pi(b,p_i^{n_i-1})| \leq |B|/(|B|,p_i^{n_i}).$$

Combining this with (5.4), we see that for every $b\in B$ we must have $B\cap\Pi(b,p_i^{n_i-1}) = B\cap\Pi(b,p_i^{n_i})$. This, again, clearly implies the condition (II) of Lemma 5.6. □

6. Classification results

6.1. **Classification results.** We now state our results on the classification of tilings with three prime factors and the (T2) property for such tilings. From now on, we restrict our attention to tilings $A\oplus B=\mathbb{Z}_M$, where $M=p_i^{n_i}p_j^{n_j}p_k^{n_k}$ has three distinct prime factors. Our main results require the additional assumption that

$$n_i = n_j = n_k = 2 \text{ and } |A| = |B| = p_ip_jp_k, \tag{6.1}$$

but some of our intermediate results are also valid without (6.1).

As in [25], we start with the assumption that $\Phi_M|A$. (Since Φ_M divides at least one of $A(X)$ and $B(X)$ by (1.3), we may always assume this after interchanging A and B if necessary.) This implies structure results for A on grids $\Lambda(x,D(M))$ for every $x\in\mathbb{Z}_M$.

Let $\Lambda:=\Lambda(a,D(M))$ for some $a\in A$, so that $A\cap\Lambda$ is nonempty. By the classic results on vanishing sums of roots of unity [3], [36], [37], [39], [33], [29], Φ_M divides $A\cap\Lambda$ if and only if $A\cap\Lambda$ is a linear combination of M-fibers with integer coefficients. In other words,

$$(A\cap\Lambda)(X) = \sum_{\nu\in\{i,j,k\}} Q_\nu(X)F_\nu(X),$$

where Q_i,Q_j,Q_k are polynomials with integer coefficients depending on both A and Λ.

In [25], we used this to develop a classification of sets $A\cap\Lambda$, where A is a finite tile, $\Phi_M|A$, and Λ is a $D(M)$-grid. One possibility is that A is M-fibered on each such grid Λ, so that $(A\cap\Lambda)(X)=Q_\Lambda(X)F_{\nu(\Lambda)}(X)$ for some $\nu(\Lambda)\in\{i,j,k\}$, possibly depending on Λ. If A is fibered on all $D(M)$-grids in the same direction, so that $\nu(\Lambda)$ can be chosen independent of Λ, the conditions of Theorem 5.2 are satisfied and we may use Corollary 5.3 to conclude that (T2) holds for both A and B. However, it is also possible for $A\cap\Lambda$ to be fibered in different directions on different grids Λ. Additionally, there may exist grids Λ such that $A\cap\Lambda$ is not fibered. This can happen if $A\cap\Lambda$ contains nonintersecting M-fibers in two or three different directions, or if some of the polynomials Q_i,Q_j,Q_k have negative coefficients, resulting in cancellations between fibers in different directions.

Our classification and (T2) results are summarized in Theorems 6.1 and 6.2 below. Both theorems were proved in [25] with the additional assumption that M is odd. In this paper, we prove that the same conclusions hold when M is even.

Theorem 6.1. *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Assume that $|A| = |B| = p_i p_j p_k$, $\Phi_M | A$, and there exists a $D(M)$-grid Λ such that $A \cap \Lambda$ is nonempty and is not M-fibered in any direction. Assume further, without loss of generality, that $0 \in \Lambda$. Then $A^\flat = \Lambda$, and the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$ via fiber shifts. By Corollary 2.2, both A and B satisfy (T2).*

Theorem 6.2. *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Assume that $|A| = |B| = p_i p_j p_k$, $\Phi_M | A$, and that for every $a \in A$, the set $A \cap \Lambda(a, D(M))$ is M-fibered in at least one direction (possibly depending on a).*

(I) *Suppose that there exists an element $a_0 \in A$ such that*

$$a_0 * F_\nu \subset A \quad \forall \nu \in \{i, j, k\}. \tag{6.2}$$

Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$ via fiber shifts, where $\Lambda := \Lambda(a_0, D(M))$. By Corollary 2.2, both A and B satisfy (T2).

(II) *Assume that no $a_0 \in A$ satisfies (6.2). Then at least one of the following holds.*

- *We have $A \subset \Pi(a, p_\nu)$ for some $a \in A$ and $\nu \in \{i, j, k\}$. By Theorem 5.1, both A and B satisfy (T2).*
- *There exists a $\nu \in \{i, j, k\}$ such that (possibly after interchanging A and B) the conditions of Theorem 5.2 are satisfied in the p_ν direction. By Corollary 5.3, both A and B satisfy (T2).*

We will provide a more detailed breakdown of the case (II) of Theorem 6.2 in Theorem 10.1, after the appropriate terminology has been introduced.

6.2. **Outline of the proof.** The general scheme of the proof of Theorems 6.1 and 6.2 is similar to that in [25] for odd M. We will take advantage of the results already proved in [25] where possible, and use the methods and technical tools developed there. In the outline below, we describe the new contributions of this paper and explain how they fit into the framework of [25].

We assume that $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$, $|A| = |B| = p_i p_j p_k$, and $\Phi_M | A$. Our proof splits into two parts according to the fibering properties of A.

Assume first that there exists a $D(M)$-grid Λ such that $A \cap \Lambda$ is not M-fibered in any direction. In Propositions 5.2 and 5.5 in [25], we proved that $A \cap \Lambda$ must then contain at least one of two special structures, either *diagonal boxes* or an *extended corner*. The latter case was resolved in [25, Theorem 8.1], for both odd and even M.

We now come to the case when M is even and $A \cap \Lambda$ contains diagonal boxes for some $D(M)$-grid Λ. As a preliminary reduction, we are able to adapt the proof of Propositions 5.2 and 6.1 in [25] to prove that $A \cap \Lambda$ must in fact consist of the diagonal boxes and possibly additional M-fibers in one or more directions (Proposition 8.5; see also the remarks after the proposition). We will also need the classification of unfibered structures with $\{m : D(M) | m | M\} \not\subset \mathrm{Div}(A)$, developed in [25, Section 6]. The relevant results are summarized in Section 8.2. They will be needed both in the unfibered case currently under consideration, and in the fibered case where they will be applied on lower scales.

We resolve the diagonal boxes case in Section 9. Our main result in this regard in Theorem 9.1, stating that if $A \cap \Lambda$ contains diagonal boxes, then A is T2-equivalent via fiber shifts to Λ. In particular, (T2) holds for both A and B.

The fiber shifting method was already used in [25], and some of our techniques are similar. The main new issue is that we have to resolve the unfibered structures in Lemma 8.6 (ii), which are significantly more difficult than the diagonal boxes structures occurring in the odd case. For one thing, they can contain very few points. (We draw the reader's attention to the case labelled here as (DB2), when $A \cap \Lambda$ consists of just two incomplete fibers and $\text{Div}(A \cap \Lambda)$ is a three-element set.) In such cases, saturating set arguments do not work as well as they did in [25], basically because there are fewer geometric restrictions coming from (3.3). We also note that, in the even case with $p_i = 2$, we have $M/p_i \notin \text{Div}(A \cap \Lambda)$ for the diagonal boxes structures in Lemma 8.6 (ii). (For M odd, such structures are ruled out by Lemma 7.8.) This causes additional difficulties in the proof. For example, we cannot exclude *a priori* the possibility that B might be M-fibered in the p_i direction; this does turn out to be impossible, but only after a longer argument.

In the particularly difficult case (DB2), we deal with this by combining saturating set techniques with splitting arguments. Intuitively, splitting arguments require less information than saturating sets, so that they can be applied in situations where saturating set arguments would lead to a tedious consideration of multiple cases. They also provide less information. In the even case, however, this can be sufficient. We refer the reader to Lemma 9.12 and Proposition 9.13 for examples of this.

In Sections 10–13, we consider the case when $A \cap \Lambda$ is fibered in some direction (possibly depending on Λ) for each $D(M)$-grid Λ. Part (I) of Theorem 6.2 was proved already in [25] for both odd and even M. Our proof of (T2) in the remaining case (II) is a significant departure from that in [25], even though some of the technical ingredients overlap. We provide a brief overview here, with a longer discussion deferred to Section 10 after the relevant concepts and notation have been introduced.

We still prove, as an intermediate result, that only two fibering directions are allowed in this case. However, instead of splitting up the proof into cases according to the intersection properties of fibers in different directions, we organize it based on the divisibility of A and B by suitably chosen cyclotomic polynomials (Section 12). We also rely on splitting arguments in parts of the proof for the even case. This turns out to lead to significant simplifications and a more clear argument than in [25, Section 9], even though our result here covers both odd and even cases.

In our final argument in the proof of Theorem 6.2 (Section 13), we also take advantage of the relationship (developed in Section 5.2) between splitting and the slab reduction. The criteria in Lemma 5.6 and Corollary 5.7 turn out to be much more convenient to use here than working directly with the conditions of Theorem 5.2.

As in [25], our final result is restricted to the case when $M = p_i^2 p_j^2 p_k^2$, but some of our methods and intermediate results are valid under weaker assumptions. For example, our results on splitting for fibered grids in Section 11 only assume that $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ has three distinct prime factors, but not that $n_i = n_j = n_k = 2$. Ultimately, however, we do need to assume that $n_i = n_j = n_k = 2$ in order to complete the proof of (T2). In that case, a single application of either the subgroup reduction or the slab reduction brings us to a case when

(T2) can be proved using simpler methods. It is likely that new multiscale methods will be needed to go beyond this constraint.

7. Toolbox

7.1. **Divisors.** The first part of Lemma 7.1 below is Lemma 8.9 of [24]. The second part is specific to the case when $p_i = 2$.

Lemma 7.1 (**Enhanced divisor exclusion).** *Let $A \oplus B = \mathbb{Z}_M$, with $M = \prod_{\iota=1}^K p_\iota^{n_\iota}$. Let $m = \prod_{\iota=1}^K p_\iota^{\alpha_\iota}$ and $m' = \prod_{\iota=1}^K p_\iota^{\alpha'_\iota}$, with $0 \leq \alpha_\iota, \alpha'_\iota \leq n_\iota$.*

(i) Assume that at least one of m, m' is different from M, and that for every $\iota \in \{1, \dots, K\}$ we have

$$\text{either } \alpha_\iota \neq \alpha'_\iota \text{ or } \alpha_\iota = \alpha'_\iota = n_\iota. \tag{7.1}$$

Then for all $x, y \in \mathbb{Z}_M$ we have

$$\mathbb{A}_m^M[x]\, \mathbb{A}_{m'}^M[x]\, \mathbb{B}_m^M[y]\, \mathbb{B}_{m'}^M[y] = 0.$$

In other words, there are no configurations $(a, a', b, b') \in A \times A \times B \times B$ such that

$$(a - x, M) = (b - y, M) = m, \quad (a' - x, M) = (b' - y, M) = m'. \tag{7.2}$$

(ii) If $p_i = 2$ for some $i \in \{1, \dots, K\}$, then for that i, the assumption (7.1) may be replaced by

$$\text{either } \alpha_i \neq \alpha'_i \text{ or } \alpha_i = \alpha'_i \in \{n_i, n_i - 1\}, \tag{7.3}$$

and the same conclusion holds.

Proof. If we did have a configuration as in (7.2), then, under the assumption (7.1) for all ι, we would have

$$(a - a', M) = (b - b', M) = \prod_{\iota=1}^K p_\iota^{\min(\alpha_\iota, \alpha'_\iota)},$$

with the right side different from M. But that is prohibited.

If $p_i = 2$ and we assume (7.3) instead of (7.1) for $\iota = i$, then the same conclusion holds, since in this case $\alpha_i = \alpha'_i = n_i - 1$ still implies that $p_i^{n_i} | a - a'$ and $p_i^{n_i} | b - b'$. □

7.2. **Cyclotomic divisibility.** The results here are borrowed from [25, Section 4.2].

Lemma 7.2 (**Cyclotomic divisibility on grids).** *Let $A \in \mathcal{M}(\mathbb{Z}_M), M = \prod_{\iota=1}^K p_\iota^{n_\iota}$, and let $m, s | M$ with $s \neq 1$. Suppose that for every $a \in A$, Φ_s divides $A \cap \Lambda(a, m)$. Then $\Phi_s | A$.*

Lemma 7.3 (**Plane bound).** *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ and $|A| = p_i^{\beta_i} p_j^{\beta_j} p_k^{\beta_k}$. Then for every $x \in \mathbb{Z}_M$ and $0 \leq \alpha_i \leq n_i$ we have*

$$|A \cap \Pi(x, p_i^{n_i - \alpha_i})| \leq p_i^{\alpha_i} p_j^{\beta_j} p_k^{\beta_k}.$$

Corollary 7.4. *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ and $|A| = p_i^{\beta_i} p_j^{\beta_j} p_k^{\beta_k}$ with $\beta_i > 0$. Suppose that for some $x \in \mathbb{Z}_M$ and $1 \leq \alpha_0 \leq n_i$*

$$|A \cap \Pi(x, p_i^{n_i - \alpha_0})| > p_i^{\beta_i - 1} p_j^{\beta_j} p_k^{\beta_k},$$

then $\Phi_{p_i^{n_i - \alpha}} | A$ for at least one $\alpha \in \{0, \dots, \alpha_0 - 1\}$.

7.3. **Saturating sets.** The following result is [25, Lemma 4.6].

Lemma 7.5 (Flat corner). *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^{n_j} p_k^{n_k}$, $|A| = p_i p_j p_k$. Suppose that*

$$\{D(M)|m|M\} \cap \mathrm{Div}(B) = \emptyset, \tag{7.4}$$

and that A contains the following 3-point configuration: for some $x \in \mathbb{Z}_M \setminus A$ there exist $a, a_j, a_k \in A$ such that

$$(a - a_j, M) = (a_k - x, M) = M/p_j,\ (a - a_k, M) = (a_j - x, M) = M/p_k$$

Then $A_x \subset \ell_i(x)$, and the pair (A, B) has a $(1, 2)$-cofibered structure, with an M-cofiber in A at distance M/p_i^2 from x.

7.4. **Fibering lemmas.** Lemma 7.6 below is a simple version of the de Bruijn-Rédei-Schoenberg theorem for cyclic groups $\mathbb{Z}_N$ under the assumption that N has at most two distinct prime factors. This has been known in the literature, see [3], [29, Theorem 3.3]. The version here is from [25, Lemma 4.7]. Lemma 7.8 is from [25, Section 4.4].

Lemma 7.6. (Cyclotomic divisibility for 2 prime factors) *Let $M = \prod_{\iota=1}^{K} p_\iota^{n_\iota}$, and let $A \in \mathcal{M}(\mathbb{Z}_N)$, for some $N \mid M$ such that $N = p_j^{\alpha_j} p_k^{\alpha_k}$ has only two distinct prime factors. Then:*

(i) $\Phi_N|A$ if and only if A is a linear combination of N-fibers in the p_j and p_k direction with non-negative integer coefficients.

(ii) Let Λ be a $D(N)$-grid. Assume that $\Phi_N|A$, and that there exists $c_0 \in \mathbb{N}$ such that $\mathbb{A}_N^N[x] \in \{0, c_0\}$ for all $x \in \Lambda$. Then $A \cap \Lambda$ is N-fibered in either the p_j or the p_k direction.

The following special case will be used several times.

Corollary 7.7. *Let $A \subset \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Assume that $\Phi_{N_i}\Phi_{M_i}|A$, and that there exists $c_0 \in \mathbb{N}$ such that*

$$\mathbb{A}_{N_i}^{N_i}[x] \in \{0, c_0\} \text{ for all } x \in \mathbb{Z}_M. \tag{7.5}$$

(Note that (7.5) is satisfied with $c_0 = 1$ if $M/p_i \notin \mathrm{Div}(A)$, and with $c_0 = p_i$ if A is M-fibered in the p_i direction.) Then A is a union of pairwise disjoint N_i-fibers in the p_j and p_k directions, each of multiplicity c_0.

Proof. Consider A modulo N_i. Without loss of generality, we may assume that $c_0 = 1$. The assumption that $\Phi_{N_i}\Phi_{M_i}|A$ implies that A mod N_i is $\mathcal{T}$-null with respect to the cuboid type $\mathcal{T} = (N_i, (0, 1, 1), 1)$. In other words, for any $N_i/p_j p_k$-grid Λ in $\mathbb{Z}_{N_i}$, Φ_{M_i} divides $(A \cap \Lambda)(X)$. By Lemma 7.6, $A \cap \Lambda$ mod N_i is N_i-fibered in one of the p_j and p_k directions. Writing $\mathbb{Z}_{N_i}$ as a union of pairwise disjoint $N_i/p_j p_k$-grids, we get the conclusion of the corollary. □

Lemma 7.8. (Missing top difference implies fibering) *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, and let $N|M$ with $p_i p_j p_k|N$. Let $\Lambda := \Lambda(x_0, D(N))$ for some $x_0 \in \mathbb{Z}_N$. Assume that $A \subset \mathbb{Z}_N$ satisfies $\Phi_N|A$ and $\Lambda \cap A \neq \emptyset$.*

(i) Suppose that $p_i \neq 2$, and that

$$N/p_i \notin \mathrm{Div}_N(A \cap \Lambda).$$

Then $A\cap\Lambda$ is N-fibered in one of the p_j and p_k directions. In particular, if $N/p_i \notin \mathrm{Div}_N(A)$, then $A\cap\Lambda$ is N-fibered in one of the p_j and p_k directions for every $D(N)$-grid Λ.

(ii) Suppose that $N/p_i, N/p_j \notin \mathrm{Div}_N(A\cap\Lambda)$. Then $A\cap\Lambda$ is N-fibered in the p_k direction.

8. Structure on unfibered grids

Throughout this section, we will use the following notation. Let $M = p_i^{n_i}p_j^{n_j}p_k^{n_k}$, and let Λ be a fixed $D(M)$-grid such that $A\cap\Lambda\neq\emptyset$. We identify Λ with $\mathbb{Z}_{p_i}\oplus\mathbb{Z}_{p_j}\oplus\mathbb{Z}_{p_k}$, and represent each point $x\in\Lambda$ as $(\lambda_i x, \lambda_j x, \lambda_k x)$ in the implied coordinate system. Assume that $A\cap\Lambda$ is not M-fibered in any direction (we will call such grids *unfibered*). In Sections 5 and 6 of [25], we classified the types of structure that A may have on such grids. Below, we outline those results and identify the cases that were already resolved in [25] for all M as above, including the even case. The remaining cases with M even will be resolved here in Section 9.

8.1. **Basic structure results.** We start with the case when $A\cap\Lambda$ is a union of pairwise disjoint fibers. Since we are assuming that $A\cap\Lambda$ is unfibered, it must contain fibers in at least two different directions. In that case, $A\cap\Lambda$ contains the following structure.

Definition 8.1. [25, Definition 5.4] *Suppose that $A\subset\mathbb{Z}_M$, and let Λ be a $D(M)$-grid.*

(i) We say that $A\cap\Lambda$ contains a p_i corner *if there exist $a, a_i\in A\cap\Lambda$ with $(a-a_i, M) = M/p_i$ satisfying*

$$A\cap(a*F_j*F_k) = a*F_j, \quad A\cap(a_i*F_j*F_k) = a_i*F_k.$$

(ii) We say that $A\cap\Lambda$ contains a p_i extended corner *if there exist $a, a_i\in A\cap\Lambda$ such that $(a-a_i, M) = M/p_i$ and*

- *$A\cap(a*F_j*F_k)$ is M-fibered in the p_j direction but not in the p_k direction,*
- *$A\cap(a_i*F_j*F_k)$ is M-fibered in the p_k direction but not in the p_j direction.*

Proposition 8.2. [25, Proposition 5.5] *Let $D = D(M)$, and let Λ be a D-grid. Assume that $A\cap\Lambda$ is a union of disjoint M-fibers, but is not fibered in any direction. Then $A\cap\Lambda$ contains a p_ν extended corner for some $\nu\in\{i,j,k\}$.*

This case was resolved entirely in [25], including the case when M is even.

Theorem 8.3. [25, Theorem 8.1] *Assume that $A\oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i}p_j^2p_k^2$, $|A| = p_ip_jp_k$, and $\Phi_M|A$. Moreover, assume that A contains a p_i extended corner in the sense of Definition 8.1 (ii) on a $D(M)$-grid Λ. Then the tiling $A\oplus B=\mathbb{Z}_M$ is T2-equivalent to $\Lambda\oplus B = \mathbb{Z}_M$ via fiber shifts. By Corollary 2.2, both A and B satisfy (T2).*

It remains to consider the case when $A\cap\Lambda$ is not a union of disjoint M-fibers. This is the case that was not resolved in [25] for even M, hence we need to do additional work here. However, a version of the preliminary structure results in [25, Proposition 5.2 and 6.1] still applies.

Definition 8.4. [25, Definition 5.1] *Let $A\subset\mathbb{Z}_M$. We say that $A\cap\Lambda$* contains diagonal boxes *if there are nonempty sets $I\subset\mathbb{Z}_{p_i}$, $J\subset\mathbb{Z}_{p_j}$, $K\subset\mathbb{Z}_{p_k}$, such that*

$$I^c := \mathbb{Z}_{p_i}\setminus I,\ J^c := \mathbb{Z}_{p_j}\setminus J,\ K^c := \mathbb{Z}_{p_k}\setminus K$$

are also nonempty, and

$$(I \times J \times K) \cup (I^c \times J^c \times K^c) \subset A \cap \Lambda.$$

Proposition 8.5. *Let $A \subset \mathbb{Z}_M$. (We do not require A to be a tile.) Assume that $\Phi_M \mid A$, and that $p_\nu = 2$ for some $\nu \in \{i, j, k\}$. Let Λ be a $D(M)$-grid such that $A \cap \Lambda$ is not a union of disjoint M-fibers. Then $A \cap \Lambda$ is a union of one set of diagonal boxes*

$$A_1 = (I_1 \times J_1 \times K_1) \cup (I_1^c \times J_1^c \times K_1^c),$$

where $I_1 \subset \mathbb{Z}_{p_i}$, $J_1 \subset \mathbb{Z}_{p_j}$, $K_1 \subset \mathbb{Z}_{p_k}$ are non-empty sets such that $I_1^c := \mathbb{Z}_{p_i} \setminus I_1$, $J_1^c := \mathbb{Z}_{p_j} \setminus J_1$, $K_1^c := \mathbb{Z}_{p_k} \setminus K_1$ are also non-empty, and possibly additional M-fibers in one or more directions, disjoint from A_1 and from each other.

Note the difference between the statement of Proposition 8.5 here and Propositions 5.2 and 6.1 in [25]. In the even case considered here, the complement of A_1 in $A \cap \Lambda$ (if nonempty) must be a disjoint union of M-fibers. No such claim is made in the odd case in [25, Proposition 5.2]. The conclusions of Proposition 8.5 are identical to those of [25, Proposition 6.1] for odd M. However, Proposition 6.1 in [25] requires stronger assumptions: we must assume, in addition, that $\{D(M)|m|M\} \not\subset \mathrm{Div}(A \cap \Lambda)$.

Proof. Let Λ be the $D(M)$-grid satisfying the assumptions of the lemma. We first claim that $A \cap \Lambda$ must contain diagonal boxes. To see this, we follow the proof of [25, Proposition 5.2] up to Claim 3, then note that the condition (5.4) of [25] cannot be satisfied when M is even, hence the proof is complete at that point.

Let A_1 be the set of diagonal boxes thus obtained. By the same argument as in the proof of Proposition 6.1 in [25], we see that $(A \cap \Lambda) \setminus A_1$ is a union of disjoint M-fibers in one or more directions.

Since our assumptions here differ from those of [25, Propositions 5.2 and 6.1], a few remarks on this are in order. Claims 1-3 in the proof of [25, Proposition 5.2] do not require A to be a tile. In [25, Proposition 6.1], we assume in (6.1) that $\{m : D(M)|m|M\} \not\subset \mathrm{Div}(A)$. The purpose of that assumption is to ensure that, in the case under consideration, the proof of [25, Proposition 5.2] can be halted after Claim 3 (see [25, (5.5)]). In our case, the same purpose is served by the assumption that M is even, so that the divisor assumption is not needed. Other than that, the proof of [25, Proposition 6.1] can be repeated verbatim here. □

Remark 8.1. *Proposition 8.5 is only stated for sets $A \subset \mathbb{Z}_M$. However, if we assume instead that $A \in \mathcal{M}(\mathbb{Z}_N)$ for some $N|M$, and that*

$$\mathbb{A}_N^N[x] \in \{0, c_0\} \text{ for some } c_0 \in \mathbb{N} \tag{8.1}$$

for all $x \in \Lambda$ (i.e., $A \cap \Lambda$ is a multiset of constant multiplicity c_0 mod N), the same argument applies except that the diagonal boxes and fibers in the conclusion also have multiplicity c_0.

8.2. **Special unfibered structures.** We will need to pay special attention to the case when $\Phi_N|A$ for some $N|M$, yet there exists a $D(N)$-grid Λ such that $\mathrm{Div}_N(A \cap \Lambda)$ does not include some of the "top differences" on that scale. We now review the results of [25] in this case. These results will be applied both to the tiling set A in a tiling $A \oplus B = \mathbb{Z}_M$, on the scale M, and to subsets of A (which do not need to be tiling complements) on various scales. Therefore, in this section we do not require A to be a tile. Since our reworking of the fibered grids case applies to both the odd and even cases, we include the odd case results in Lemma 8.7.

We will work under the following set of assumptions for unfibered grids with missing top differences.

Assumption (UF). Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. Assume that $N|M$ satisfies $N = M/p_i^{\alpha_i} p_j^{\alpha_j} p_k^{\alpha_k}$ with $\alpha_\iota < n_\iota$ for all $\iota \in \{i, j, k\}$, so that any $D(N)$-grid is 3-dimensional. Let Λ be a $D(N)$-grid such that $A \cap \Lambda \neq \emptyset$. Assume that:

- $A \in \mathcal{M}(\mathbb{Z}_N)$ satisfies $\Phi_N|A$
- there exists $c_0 \in \mathbb{N}$ such that (8.1) holds for all $x \in \Lambda$,
- $A \cap \Lambda$ is not fibered in any direction,

and that

$$\{m : D(N)|m|N\} \not\subset \mathrm{Div}_N(A \cap \Lambda).$$

Lemma 8.6. [25, Lemmas 6.4 and 6.5] *Assume (UF), with $2|M$. Then, possibly after a permutation of $\{i, j, k\}$, one of the following must hold.*

(i) We have $\{D(N)|m|N\} \setminus \mathrm{Div}_N(A \cap \Lambda) = \{N/p_ip_j\}$, and $A \cap \Lambda$ has a p_k corner structure in the sense of Definition 8.1 (i).

(ii) We have $p_k = 2$ and $N/p_k \notin \mathrm{Div}_N(A \cap \Lambda)$. Moreover, there is a pair of diagonal boxes

$$A_0 = (I \times J \times K) \cup (I^c \times J^c \times K^c) \subset \Lambda,$$

as in Definition 8.4, such that for all $z \in A \cap \Lambda$ we have

$$\mathbb{A}_N^N[z] = c_0 \mathbf{1}_{A_0}(z).$$

Thus, $A \cap \Lambda$ is a set of diagonal boxes with multiplicity c_0, with no additional fibers in the same grid.

Lemma 8.7. [25, Lemmas 6.2 and 6.3] *Assume (UF) with M odd. Then, possibly after a permutation of $\{i, j, k\}$, one of the following must hold.*

(i) *(p_i-**full plane**) We have*

$$\{D(N)|m|N\} \setminus \mathrm{Div}_N(A \cap \Lambda) = \{N/p_ip_j, N/p_ip_k\},$$

and there exists $x \in \mathbb{Z}_M \setminus A$ such that

$$\mathbb{A}_{N/p_i}^N[x] = c_0\phi(p_i),\ \mathbb{A}_{N/p_jp_k}^N[x] = c_0\phi(p_jp_k),$$

$$\mathbb{A}_m^N[x] = 0 \text{ for all } m \in \{D(N)|m|N\} \setminus \{N/p_i, N/p_jp_k\}.$$

(ii) *(p_k-**corner**, cf. Definition 8.1 (i).) We have*

$$\{D|m|N\} \setminus \mathrm{Div}_N(A \cap \Lambda) = \{N/p_ip_j\}, \tag{8.2}$$

and for each $x \in \Lambda$, the set $A \cap \Lambda \cap \Pi(x, p_k^{n_k - \alpha_k})$ is either empty or consists of a single N-fiber in one of the p_i or p_j directions. Since $A \cap \Lambda$ is not fibered, there has to be at least one of each.

(*iii*) **(p_k-almost corner)** *We have (8.2), and $A\cap\Lambda$ has the following structure. There exist $x_0, x_1, \dots, x_{\phi(p_k)} \in \mathbb{Z}_N$ with $(x_l - x_{l'}, N) = N/p_k$ for $l \neq l'$, and two disjoint sets $\mathcal{L}_i, \mathcal{L}_j \subset \mathbb{Z}_{p_k}$ satisfying $|\mathcal{L}_i|, |\mathcal{L}_j| > 1$ and $\mathcal{L}_i \cup \mathcal{L}_j = \{0, 1, \dots, \phi(p_k)\}$, such that for all $z \in \Lambda$ we have*

$$\mathbb{A}_N^N[z] = \begin{cases} c_0 & \text{if } (z - x_l, N) = N/p_i \text{ for some } l \in \mathcal{L}_i \\ & \quad \text{or } (z - x_l, N) = N/p_j \text{ for some } l \in \mathcal{L}_j \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathbb{A}_N^N[x_l] = 0$ and $\mathbb{A}_{N/p_i}^N[x_l] = c_0\phi(p_i)$ for all $l \in \mathcal{L}_i$, and similarly with i and j interchanged.

9. Resolving diagonal boxes

Our main theorem for the diagonal boxes case is as follows.

Theorem 9.1. *Let $A \oplus B = \mathbb{Z}_M$ be a tiling, where $M = p_i^2 p_j^2 p_k^2$ is even and $|A| = |B| = p_i p_j p_k$, and assume that $\Phi_M \mid A$. Let $D = D(M)$, and let Λ be a $D(M)$-grid such that $A \cap \Lambda \neq \emptyset$. Assume further that $A \cap \Lambda$ is not a disjoint union of M-fibers. Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$ via fiber shifts. Thus Λ is a translate of $A^\flat$, and by Corollary 2.2, both A and B satisfy (T2).*

We begin the proof with a few reductions. Let A and Λ be as in Theorem 9.1, so that in particular $A \cap \Lambda$ is not a disjoint union of M-fibers. By Proposition 8.5, $A \cap \Lambda$ is a union of one pair of diagonal boxes and possibly additional M-fibers, disjoint from the boxes and from each other. We fix that pair of diagonal boxes, and denote it

$$A_1 = (I \times J \times K) \cup (I^c \times J^c \times K^c).$$

The proof splits further into cases, depending on the dimensions of boxes and on whether we have

$$\{m :\ D(M)|m|M\} \subset \mathrm{Div}(A \cap \Lambda). \tag{9.1}$$

If (9.1) fails, by Lemma 8.6 we must have

$$A \cap \Lambda = A_1. \tag{9.2}$$

We note that the corner structure in Lemma 8.6 (i) is a union of disjoint M-fibers, therefore does not fall under the purview of Theorem 9.1. This case was already resolved in [25, Theorem 8.1], stated here as Theorem 8.3.

We claim that it suffices to consider the following sets of assumptions.

Case (DB1): The tiling $A \oplus B$ satisfies the assumptions of Theorem 9.1. Additionally, we have $p_i = 2$, (9.2) holds, and $\min(|J^c|, |K^c|) \geq 2$.

Case (DB2): The tiling $A \oplus B$ satisfies the assumptions of Theorem 9.1. Additionally, we have $p_i = 2$, (9.2) holds, and $|J^c| = |K| = 1$.

Case (DB3): The tiling $A \oplus B$ satisfies the assumptions of Theorem 9.1. Moreover, $p_i = 2$, (9.1) holds, and $A \cap \Lambda$ is a union of a pair of diagonal boxes A_1 and one or more M-fibers, disjoint from the boxes and from each other.

Indeed, assume that M is even, with $p_i = 2$. Without loss of generality, we have $I = \{0\}, I^c = \{1\}$.

- If (9.1) holds, we cannot have $A \cap \Lambda = A_1$, since $M/p_i \notin \mathrm{Div}(A_1)$. Hence $(A \cap \Lambda) \setminus A_1$ is a nonempty union of M-fibers, and we are in the case (DB3).
- Suppose now that (9.1) fails. Then (9.2) holds by Lemma 8.6 (ii). Moreover, at least one of the sets from each pair J, J^c and K, K^c must have cardinality greater than 1. If $\min(|J|, |K|) \geq 2$ or $\min(|J^c|, |K^c|) \geq 2$, we are in the case (DB1), and if either $|J^c| = |K| = 1$ or $|J| = |K^c| = 1$, we are in the case (DB2), possibly after relabelling the sets.

The case (DB3) was resolved in [25, Corollary 7.3].

Proposition 9.2. *(Special case of* [25, Corollary 7.3]*) Assume that (DB3) holds. Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$. Consequently, A and B both satisfy T2.*

We will proceed to resolve the cases (DB1) and (DB2) in Sections 9.1 and 9.2, respectively. We will be able to reduce these situations to either (DB3) or an extended corner structure. By Theorem 8.3 and Proposition 9.2, this implies the conclusion of the theorem.

9.1. **Case (DB1).** Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling satisfying the assumptions of Theorem 9.1. Let Λ be the $D(M)$-grid provided by the assumption of the theorem. Additionally, we assume that $p_i = 2$, (9.2) holds, and

$$\min(|J^c|, |K^c|) \geq 2. \tag{9.3}$$

In particular, (9.2) implies that every point $x \in \mathbb{Z}_M$ such that

$$x \in I^c \times J \times K \tag{9.4}$$

satisfies $x \notin A$ and $\mathbb{A}_{M/p_j}[x] = \mathbb{A}_{M/p_k}[x] = 0$.

The proof of Theorem 9.1 is based on saturating set considerations, leading to the T2-equivalence via fiber shifting as stated in the theorem. Compared to the corresponding case with M odd, an additional difficulty is that the assumption (DB1) does not exclude *a priori* the possibility that $M/p_i \in \mathrm{Div}(B)$. It turns out, however, that $M/p_i \in \mathrm{Div}(B)$ implies that B must in fact be M-fibered in the p_i direction. By Lemma 9.8, this is incompatible with the structure of A provided by (DB1). The remaining cases are covered by Lemma 9.7.

Lemma 9.3. *Assume (DB1), and let x satisfy (9.4). Then for every $b \in B$ we have exactly one of the following:*

$$A_{x,b} \subset \ell_j(x), \ \textit{with } \mathbb{A}_{M/p_j^2}[x]\mathbb{B}_{M/p_j^2}[b] = \phi(p_j^2), \tag{9.5}$$

$$A_{x,b} \subset \ell_k(x), \ \textit{with } \mathbb{A}_{M/p_k^2}[x]\mathbb{B}_{M/p_k^2}[b] = \phi(p_k^2), \tag{9.6}$$

$$A_{x,b} \subset \ell_i(x) \ \textit{with } \mathbb{A}_{M/p_i}[x]\mathbb{B}_{M/p_i}[b] = \phi(p_i) \tag{9.7}$$

Furthermore:

- *If (9.5) holds for some $b \in B$, then the product $\langle \mathbb{A}[x], \mathbb{B}[b] \rangle$ is saturated by a $(1,2)$-cofiber pair in the p_j direction, with the A-cofiber at distance M/p_j^2 from x and the B-fiber rooted at b. The same is true for (9.6), with k and j interchanged.*

- *If (9.7) holds for some $b \in B$, then the product $\langle \mathbb{A}[x], \mathbb{B}[b] \rangle$ is saturated by a $(0,1)$-cofiber pair in the p_i direction, i.e. $\mathbb{A}_{M/p_i}[x] = \mathbb{B}_{M/p_i}[b] = 1$.*

Proof. By (DB1) and (9.4), we have $A_x \subset \Pi(x, p_i^2) \cup \Pi(a, p_i^2)$, where $a \in A$ satisfies $(a - x, M) = M/p_i$. Moreover,

$$A_x \subset \bigcap_{a':(x-a',M)=M/p_jp_k} \mathrm{Bispan}(x, a') = \Pi(x, p_j^2) \cup \Pi(x, p_k^2),$$

thus

$$A_x \subset \ell_j(x) \cup \ell_k(x) \cup \ell_j(a) \cup \ell_k(a). \tag{9.8}$$

By (9.3), we also have

$$\{M/p_jp_k \mid m \mid M\} \subset \mathrm{Div}(A). \tag{9.9}$$

If $\mathbb{B}_{M/p_i}[b] > 0$, then $\mathbb{A}_{M/p_i}[x]\mathbb{B}_{M/p_i}[b] = \phi(p_i) = 1$ and we are in the case (9.7). Assume therefore, for the rest of the proof, that

$$\mathbb{B}_{M/p_i}[b] = 0. \tag{9.10}$$

We start by claiming that

$$\text{at most one of } A_{x,b} \cap \ell_j(x) \text{ and } A_{x,b} \cap \ell_j(a) \text{ is non-empty.} \tag{9.11}$$

Indeed, suppose that $A_{x,b} \cap \ell_j(x) \neq \emptyset$. By (9.9), this is only possible if

$$\mathbb{A}_{M/p_j^2}[x]\mathbb{B}_{M/p_j^2}[b] > 0. \tag{9.12}$$

Suppose that we also have $A_{x,b} \cap \ell_j(a) \neq \emptyset$, then

$$\mathbb{A}_{M/p_i}[x]\mathbb{B}_{M/p_i}[b] + \mathbb{A}_{M/p_ip_j}[x]\mathbb{B}_{M/p_ip_j}[b] + \mathbb{A}_{M/p_ip_j^2}[x]\mathbb{B}_{M/p_ip_j^2}[b] > 0.$$

The first term is 0 by (9.10). The second term cannot be non-zero concurrently with (9.12), by Lemma 7.1. For the third term to be non-zero, we would need $\mathbb{A}_{M/p_ip_j^2}[x] = \mathbb{A}_{M/p_j^2}[a] > 0$, but that again cannot hold concurrently with (9.12). This proves (9.11).

By Lemma 7.1, if either of the sets in (9.11) is non-empty, we have $A_{x,b} \cap (\ell_k(x) \cup \ell_k(a)) = \emptyset$. Repeating the same argument with j and k interchanged, we get that $A_{x,b}$ is in fact contained in just one of the lines in (9.8).

It remains to prove that we cannot have $A_{x,b} \subset \ell_j(a)$ or $A_{x,b} \subset \ell_k(a)$. Suppose that $A_{x,b} \subset \ell_j(a)$. Then, using (9.10) and that $p_i = 2$, we get

$$\begin{aligned} 1 &= \frac{1}{\phi(p_ip_j)} \mathbb{A}_{M/p_ip_j}[x|\ell_j(a)]\mathbb{B}_{M/p_ip_j}[b] + \frac{1}{\phi(p_ip_j^2)} \mathbb{A}_{M/p_ip_j^2}[x|\ell_j(a)]\mathbb{B}_{M/p_ip_j^2}[b] \\ &= \frac{1}{\phi(p_j)} \mathbb{A}_{M/p_j}[a]\mathbb{B}_{M/p_j}[y] + \frac{1}{\phi(p_j^2)} \mathbb{A}_{M/p_j^2}[a]\mathbb{B}_{M/p_j^2}[y], \end{aligned}$$

where $y \in \mathbb{Z}_M \setminus B$ is the unique element such that $(b - y, M) = M/p_i$. If both expressions in the last sum were nonzero, we would have $\mathbb{A}_{M/p_j^2}[a]\mathbb{B}_{M/p_j}[y]\mathbb{B}_{M/p_j^2}[y] > 0$, hence $M/p_j^2 \in \mathrm{Div}(A) \cap \mathrm{Div}(B)$, which is a contradiction. This means that either

$$\phi(p_j) = \mathbb{A}_{M/p_j}[a]\mathbb{B}_{M/p_j}[y] \tag{9.13}$$

or

$$\phi(p_j^2) = \mathbb{A}_{M/p_j^2}[a]\mathbb{B}_{M/p_j^2}[y]. \tag{9.14}$$

Suppose that (9.14) holds. Since M/p_j, $M/p_j^2 \in \mathrm{Div}(A)$, we must have $\mathbb{B}_{M/p_j^2}[y] = 1$, so that $\mathbb{A}_{M/p_j^2}[a] = \phi(p_j^2)$. But this implies $|A \cap \Pi(a, p_k^2)| \geq \mathbb{A}_{M/p_j^2}[a] + \mathbb{A}_M[a] \geq \phi(p_j^2) + 1 > p_i p_j$, which contradicts Lemma 7.3.

Finally, we prove (9.13) cannot hold. Assume the contrary. Then, since $M/p_j \notin \mathrm{Div}(B)$, we must have $\mathbb{B}_{M/p_j}[y] = 1$ and therefore

$$\mathbb{A}_{M/p_j}[a] = \phi(p_j).$$

However, this contradicts the assumption (9.2).

The proof that we cannot have $A_{x,b} \subset \ell_k(a)$ is identical. The lemma follows. □

Corollary 9.4. *Assume (DB1), and fix $b \in B$. Suppose that $A_{x,b} \subset \ell_\nu(x)$, where $\nu \in \{i, j, k\}$ and x satisfies (9.4). Then $A_{x',b} \subset \ell_\nu(x')$ for all x' satisfying (9.4).*

Proof. Assume first that $\mathbb{B}_{M/p_i}[b] = 1$. Since $\mathbb{A}_{M/p_i}[x] = 1$ for all x satisfying (9.4), it follows that (9.7) holds for all such x, as required.

We now prove the corollary with $\nu = j$. Assume that $A_{x,b} \subset \ell_j(x)$, so that (9.7) holds. Clearly, (9.7) also holds for all x' such that $(x' - x, M) = M/p_j$, so that

$$A_{x_j,b} \subset \ell_j(x_j) \quad \forall x_j \in x * F_j. \tag{9.15}$$

We also note that

$$M/p_i p_j^2 \in \mathrm{Div}(\ell_j(x), a) \subset \mathrm{Div}(A) \tag{9.16}$$

where a satisfies $(x - a, M) = M/p_i$.

If $|K| = 1$, the above argument already proves the lemma. Assume now that $|K| > 1$, and let x'' satisfy (9.4) and $(x'' - x_j, M) = M/p_k$ for some $x_j \in x * F_j$. By (9.15), we have $\mathbb{B}_{M/p_j^2}[b] > 0$, and by (9.16) we have $M/p_i p_j^2 \notin \mathrm{Div}(B)$. It follows that $\mathbb{B}_{M/p_i}[b] = 0$, so that $A_{x'',b} \cap \ell_i(x'') = \emptyset$. Suppose now that $A_{x'',b} \subset \ell_k(x'')$. By the same argument as above with j and k interchanged, this would imply that $A_{x_j,b} \subset \ell_k(x_j)$, which would contradict (9.15). Hence $A_{x'',b} \subset \ell_j(x'')$, and as above, the same is true for all x''_j with $(x''_j - x'', M) = M/p_j$. This proves the lemma for $\nu = j$.

The proof for $\nu = k$ is identical, interchanging k and j. □

Lemma 9.5. *Assume that (DB1) holds. Let x satisfy (9.4). If there exists $b \in B$ for which (9.5) holds, then (9.6) does not hold for any $b' \in B$. The same holds with (9.5) and (9.6) interchanged.*

Proof. Assume, by contradiction, that there exist $b_j, b_k \in B$ such that b_j satisfies (9.5) and b_k satisfies (9.6). It follows by Corollary 9.4 that (9.5) with $b = b_j$ and (9.6) with $b = b_k$ hold for all $x' \in I^c \times J \times K$. Thus $A \cap \Pi(x, p_i^2)$ contains $I^c \times J^c \times K^c$, as well as $|K|$ M-fibers in the p_j direction and $|J|$ M-fibers in the p_k direction, all disjoint from each other. We get

$$\begin{aligned} |A \cap \Pi(x, p_i^2)| &\geq (p_j - |J|)(p_k - |K|) + p_k|J| + p_j|K| \\ &= p_j p_k + |J||K| > p_j p_k. \end{aligned}$$

This contradicts Lemma 7.3, and completes the proof of the lemma. □

Lemma 9.6. *Assume (DB1) holds, and that (9.5) holds for some x satisfying (9.4) and $b \in B$. Then, for every $x' \in I^c \times J \times K^c$,*

$$A_{x',b} \subset \ell_i(x'), \text{ with } \mathbb{A}_{M/p_i^2}[x'] = p_i,\ \mathbb{B}_{M/p_i^2}[b] = \phi(p_i). \tag{9.17}$$

Moreover, (9.17) holds for all $b \in B$, and B is N_i-fibered in the p_i direction.

If (9.6) holds for some x satisfying (9.4) and $b \in B$, then the same conclusion is true with j and k interchanged.

Proof. Suppose that (9.5) holds for some x satisfying (9.4), and for some $b \in B$. By Corollary 9.4 and Lemma 9.5, (9.5) holds for the same $b \in B$ with x replaced by any other element x'' satisfying (9.4).

Let $x' \in I^c \times J \times K^c$. By the above argument, we may assume that

$$(x - x', M) = M/p_k.$$

Denote by a the unique element in $I \times J \times K$ satisfying

$$(x - a, M) = M/p_i, (x' - a, M) = M/p_ip_k.$$

Claim 1. Let $b \in B$. If b satisfies either $A_{x,b} \subset \ell_i(x)$ or (9.5), then

$$A_{x',b} \subset \ell_i(x') \cup \ell_i(x). \tag{9.18}$$

Proof. Consider the saturating set $A_{x',b}$. By (9.3) we have $\mathbb{A}_{M/p_j}[x'] \geq 2$, so that $A_{x',b} \subset \Pi(x', p_j^2)$. Additionally, applying (3.2) to x and x', and using that $A_{x,b} \subset \ell_\mu(x)$ for some $\mu \in \{i, j\}$, we get

$$A_{x',b} \subset \Pi(x', p_k^2) \cup \Pi(x, p_k^2).$$

Hence (9.18) follows. □

Claim 2. Let $b \in B$ satisfy (9.5). Then

$$A_{x',b} \subset \ell_i(x'). \tag{9.19}$$

Proof. Clearly, if $|K| > 1$ then $A_{x',b} \cap \ell_i(x) \neq \emptyset$ implies $A_{x'',b} \cap \ell_i(x) \neq \emptyset$ for all x'' satisfying (9.4) with $(x - x'', M) = M/p_k$. But this would contradict the fact that x'' satisfies (9.5). We are left with the case $|K| = 1$. Assume, for contradiction, that

$$A_{x',b} \cap \ell_i(x) \neq \emptyset. \tag{9.20}$$

We first claim that (9.20) implies

$$A_{x',b} \subset \ell_i(x). \tag{9.21}$$

Indeed, (9.20) implies that either $\mathbb{A}_{M/p_i^2p_k}[x'|\ell_i(x)] = \mathbb{A}_{M/p_i^2}[a] > 0$, or else $a \in A_{x',b}$ and $\mathbb{B}_{M/p_ip_k}[b] > 0$. If the former holds, then $M/p_i^2 \in \text{Div}(A)$; since $\mathbb{A}_{M/p_i}[x'] = 0$ by (9.2), this implies that $A_{x',b} \cap \ell_i(x') = \emptyset$ as claimed. Assume now that $a \in A_{x',b}$. Then the failure of (9.21) would imply that $M/p_i^2p_k \in \text{Div}(a, \ell_i(x)) \subset \text{Div}(A)$; on the other hand, we have $M/p_i^2p_k \in \text{Div}(\ell_i(b), b'') \subset \text{Div}(B)$, where $(b - b'', M) = M/p_ip_k$. The claim follows.

With (9.21) in place, we now have

$$1 = \frac{1}{\phi(p_k)} \mathbb{A}_{M/p_ip_k}[x'|\ell_i(x)]\mathbb{B}_{M/p_ip_k}[b] + \frac{1}{\phi(p_i^2p_k)} \mathbb{A}_{M/p_i^2p_k}[x'|\ell_i(x)]\mathbb{B}_{M/p_i^2p_k}[b]. \tag{9.22}$$

By (9.2), we have $\mathbb{A}_{M/p_ip_k}[x'|\ell_i(x)] = 1$, and since

$$M/p_k \in \text{Div}(A), \tag{9.23}$$

we have $\mathbb{B}_{M/p_ip_k}[b] \leq 1$. It follows that

$$\mathbb{A}_{M/p_i^2p_k}[x'|\ell_i(x)]\mathbb{B}_{M/p_i^2p_k}[b] > 0, \tag{9.24}$$

with

$$\mathbb{A}_{M/p_i^2p_k}[x'|\ell_i(x)] \leq \phi(p_i^2) \text{ and } \mathbb{B}_{M/p_i^2p_k}[b] \leq 2, \tag{9.25}$$

where the latter follows from the fact that $p_i = 2$ and (9.23). Plugging in these restrictions to (9.22), we get $\phi(p_k) \leq 3$.

It is, therefore, left to consider the case when

$$p_k = 3 \text{ and } |K| = 1.$$

We then have

$$\begin{aligned}
|A \cap \Pi(x, p_k)| &\geq |A_1| + p_j \cdot |K| \\
&= p_jp_k - p_k|J| + 2|J||K| \\
&= 3p_j - |J| \\
&> p_ip_j
\end{aligned}$$

and by Corollary 7.4

$$\Phi_{p_k^2}|A. \tag{9.26}$$

If $\mathbb{B}_{M/p_ip_k}[b] = 0$, it follows from (9.25) with $p_k = 3$ that $\mathbb{A}_{M/p_i^2p_k}[x'|\ell_i(x)] = \phi(p_i^2)$ as well. In particular, $M/p_i \in \text{Div}(A)$, hence (9.7) does not hold for any $b \in B$. Therefore (9.5) must hold for both $b_1, b_2 \in B$ with $(b - b_\mu, M) = M/p_i^2p_k$, for $\mu = 1, 2$. Since (9.5) also holds for b, and $b_1, b_2 \in \Pi(b, p_k)$, we get

$$|B \cap \Pi(b, p_k)| \geq 3p_j > p_ip_j.$$

Applying Corollary 7.4 to B, we get $\Phi_{p_k^2}|B$, contradicting (9.26).

When $\mathbb{B}_{M/p_ip_k}[b] = 1$, we denote $b' \in B$ with $(b - b', M) = M/p_ip_k$. From (9.23), it is evident that $\mathbb{B}_{M/p_i}[b'] = 0$ and so b' cannot satisfy (9.7). It follows that b' satisfies (9.5). In addition, by (9.24), there must also be $b'' \in B$ with $(b - b'', M) = M/p_i^2p_k$. Combining all of the above, we have

$$|B \cap \Pi(b, p_k)| \geq 2p_j + 1 > p_ip_j.$$

By Corollary 7.4, we get $\Phi_{p_k^2}|B$ again, contradicting (9.26). □

Claim 3. Let $b \in B$ satisfy (9.5). Then (9.17) holds.

Proof. By Claim 2, (9.19) holds. We need to prove that

$$\mathbb{A}_{M/p_i^2}[x'] = p_i, \ \mathbb{B}_{M/p_i^2}[b] = \phi(p_i). \tag{9.27}$$

Assume, by contradiction, that (9.27) does not hold. Then

$$\mathbb{A}_{M/p_i^2}[x'] = \phi(p_i), \ \mathbb{B}_{M/p_i^2}[b] = p_i \tag{9.28}$$

and

$$M/p_i, M/p_i^2 \in \text{Div}(B). \tag{9.29}$$

Let $b' \in B$ with $(b-b', M) = M/p_i^2$ so that

$$\mathbb{B}_{M/p_i}[b'] = \mathbb{B}_{M/p_i^2}[b'] = 1. \tag{9.30}$$

We show that the product $\langle \mathbb{A}[x'], \mathbb{B}[b'] \rangle$ cannot be saturated.

Consider the saturating set $A_{x',b'}$. Since $\mathbb{A}_{M/p_i}[x]\mathbb{B}_{M/p_i}[b'] = 1 = \phi(p_i)$, we have $A_{x,b'} \subset \ell_i(x)$, so that the assumptions of Claim 1 are satisfied with b replaced by b'. It follows that

$$A_{x',b'} \subset \ell_i(x') \cup \ell_i(x).$$

By (9.29), if $A_{x',b'} \cap \ell_i(x)$ is nonempty, then $a \in A_{x',b'}$, hence $\mathbb{B}_{M/p_ip_k}[b'] > 0$; but this is in contradiction with (9.23) and (9.30). It follows that $A_{x',b'} \subset \ell_i(x')$. By (9.2), (9.28) and (9.30) we get

$$1 = \frac{1}{\phi(p_i^2)} \mathbb{A}_{M/p_i^2}[x']\mathbb{B}_{M/p_i^2}[b'] = 1/2.$$

This contradiction proves (9.27). ☐

Claim 4. Suppose there exists $b \in B$ satisfying (9.5). Then (9.27) holds for all $b \in B$. Consequently B is N_i-fibered in the p_i direction.

Proof. By (9.27), we have $M/p_i \in \mathrm{Div}(A)$ and $M/p_i^2 \in \mathrm{Div}(B)$. We conclude that no element $b' \in B$ satisfies (9.7), so that (9.27) holds for all $b \in B$. ☐

This ends the proof of Lemma 9.6. ☐

Lemma 9.7. *Assume that (DB1) holds, and that (9.5) holds for some x satisfying (9.4) and some $b \in B$. Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$ via fiber shifts. Therefore, the conclusions of Theorem 9.1 are satisfied.*

Proof. Assume that there exists $x \in I^c \times J \times K$ satisfying (9.4) and (9.5) for some $b \in B$. By Lemma 9.6, for every element $x' \in I^c \times J \times K^c$ the line $\ell_i(x)$ contains an M-fiber in the p_i direction, at distance M/p_i^2 from x', and B is N_i-fibered in the p_i direction. Let A' be the set obtained from A by shifting all such M-fibers, so that after these shifts we have $\{0,1\} \times J \times K^c \subset A'$. By Lemma 3.11, A' is T2-equivalent to A. Moreover, $A' \cap \Lambda$ contains both the original set of diagonal boxes A_1 and at least two (since $|K^c| \geq 2$) additional M-fibers in the p_i direction, disjoint from A_1. Therefore the tiling $A' \oplus B = \mathbb{Z}_M$ satisfies the assumption (DB3). The conclusion follows from Proposition 9.2. ☐

Since the argument above is symmetric with respect to interchanging j and k, the same conclusion holds if (9.6) holds for some x satisfying (9.4) and some $b \in B$. It remains to consider the case when (9.7) holds for all x satisfying (9.4) and all $b \in B$. This, however, means that B is M-fibered in the p_i direction. We could apply the slab reduction to B and conclude that (T2) holds in this case as well. However, with only slightly more effort, we can prove that the hypothetical tiling obtained from the slab reduction cannot actually exist, so that this case can be eliminated altogether.

Lemma 9.8. *Assume (DB1). Then B cannot be M-fibered in the p_i direction.*

Proof. Assume, for contradiction, that B is M-fibered in the p_i direction. We apply Theorem 5.2 and Corollary 5.3 to get a tiling $A' \oplus B' = \mathbb{Z}_{M'}$, where $M' = M/p_i$, $A' \equiv A \bmod M'$, and B' is obtained from B by selecting one element from each M-fiber in the p_i direction and then reducing mod M'. In particular, $|B'| = p_j p_k$. We also use $\mathbb{A}'$ and $\mathbb{B}'$ to denote the boxes associated with A' and B' in $\mathbb{Z}_{M'}$.

Let $\Lambda' = \{x' \in \mathbb{Z}_{M'} : \ p_i | a - x'\}$ for any $a \in A'_1$, where A'_1 is the reduction of A_1 mod M'. (Note that the images of both of the diagonal boxes in A_1 lie in one plane $\Pi(a, p_i)$ mod M' after this reduction.) Let $x'_j \in \Lambda' \setminus A'_1$ so that $x'_j \equiv x_j$ mod M' for some $x_j \in I \times J^c \times K$. By (9.2), we have $x'_j \notin A'$, and

$$\mathbb{A}'_{M'/p_j}[x'_j] \geq 1, \quad \mathbb{A}'_{M'/p_k}[x'_j] \geq 2. \tag{9.31}$$

In this proof, we will use A'_{x_j} and $A'_{x_j,b}$ to denote saturating sets with respect to the tiling $A' \oplus B' = \mathbb{Z}_{M'}$. We claim that

$$A'_{x_j} \subset \ell_i(x_j), \text{ with } \mathbb{B}'_{M'/p_i}[b] = 1 \text{ for all } b \in B'. \tag{9.32}$$

In particular, B' must be M'-fibered in the p_i direction, contradicting the fact that $|B'| = p_j p_k$.

We now prove (9.32). If $\mathbb{A}'_{M'/p_j}[x'_j] \geq 2$, (9.32) follows immediately from (9.31) and (3.3). If $\mathbb{A}'_{M'/p_j}[x'_j] = 1$, we have

$$A'_x \subset \ell_i(x_j) \cup \ell_i(a_j),$$

where a_j is the unique element of A' with $(a_j - x_j, M') = M'/p_j$. Let $b \in B'$. Since $M'/p_j \in \mathrm{Div}_{M'}(A')$, we have $\mathbb{B}'_{M'/p_j}[b] = 0$ and $\mathbb{B}'_{M'/p_i p_j}[b] \leq 1$, so that

$$\langle \mathbb{A}'[x'_j | \ell_i(a_j)], \mathbb{B}'[b] \rangle \leq \frac{1}{\phi(p_i p_j)} < 1.$$

Hence $A'_{x_j,b} \cap \ell_i(x_j) \neq \emptyset$. But then there must be a $b' \in B'$ such that $(b - b', M') = M'/p_i$. SInce $b \in B'$ was arbitrary, the claim follows. □

9.2. **Case (DB2).** We continue to assume that the tiling $A \oplus B = \mathbb{Z}_M$ satisfies the conditions of Theorem 9.1. Let Λ be the $D(M)$-grid as in the statement of the theorem. Additionally, we assume that $p_i = 2$ and that (9.2) holds with

$$|J^c| = |K| = 1. \tag{9.33}$$

Let $x_j, x_k \in \mathbb{Z}_M$ such that $I \times J^c \times K = \{x_j\}$ and $I^c \times J^c \times K = \{x_k\}$. By (9.2) and (9.33), we have $x_j, x_k \notin A$ and

$$(x_j - x_k, M) = M/p_i \text{ and } \frac{1}{\phi(p_j)} \mathbb{A}_{M/p_j}[x_j] = \frac{1}{\phi(p_k)} \mathbb{A}_{M/p_k}[x_k] = 1. \tag{9.34}$$

From this point, the proof is organized as follows. We prove in Lemma 9.9 that for each $b \in B$, the saturating set $A_{x_j,b}$ is contained in one of the lines $\ell_i(x_j), \ell_k(x_j), \ell_k(x_k)$, and that the same is true with j and k interchanged. We then ask which combinations of these lines can work for both of the points x_j, x_k simultaneously. Define

$$B_0 := \{b \in B : \ A_{x_j,b}, A_{x_k,b} \subset \ell_i(x_j)\},$$
$$B_1 := \{b \in B : \ A_{x_j,b} \subset \ell_k(x_j) \text{ and } A_{x_k,b} \subset \ell_j(x_k)\}.$$

(Note that $\ell_i(x_j) = \ell_i(x_k)$.) We will prove that

$$B = B_0 \text{ or } B = B_1. \tag{9.35}$$

In both cases, we will be able to use fiber shifts to prove Theorem 9.1.

Lemma 9.9. *Assume (DB2). Then, for each $b \in B$, $A_{x_j,b}$ is contained in exactly one of the lines $\ell_i(x_j), \ell_k(x_j), \ell_k(x_k)$. The same statement holds with j and k interchanged. In particular, the sets B_0 and B_1 are disjoint. Moreover, for any $b \in B$,*

$$A_{x_j,b} \subset \ell_i(x_j) \ \Leftrightarrow\ A_{x_k,b} \subset \ell_i(x_j). \tag{9.36}$$

Proof. Fix $b \in B$. By span considerations we have

$$A_{x_j,b} \subset \ell_i(x_j) \cup \ell_k(x_j) \cup \ell_k(x_k).$$

We prove first that $A_{x_j,b}$ cannot intersect both $\ell_k(x_j)$ and $\ell_k(x_k)$ simultaneously. Suppose that

$$A_{x_j,b} \cap \ell_k(x_j) \neq \emptyset. \tag{9.37}$$

Since $M/p_k \in \mathrm{Div}(A)$, we must have

$$\mathbb{A}_{M/p_k^2}[x_j]\mathbb{B}_{M/p_k^2}[b] > 0. \tag{9.38}$$

Assume furthermore that $A_{x_j,b} \cap \ell_k(x_k) \neq \emptyset$. If

$$\mathbb{A}_{M/p_ip_k^2}[x_j|\ell_k(x_k)]\mathbb{B}_{M/p_ip_k^2}[b] > 0,$$

then this together with (9.34) would imply that $M/p_k^2 \in \mathrm{Div}(A\cap\ell_k(x_k))$, contradicting (9.38). Hence $\mathbb{A}_{M/p_ip_k}[x_j|\ell_k(x_k)]\mathbb{B}_{M/p_ip_k}[b] > 0$. But by Lemma 7.1, this cannot hold concurrently with (9.38).

Next, suppose that $A_{x_j,b} \cap \ell_i(x_j) \neq \emptyset$. Since $x_k \notin A$, we must have $\mathbb{A}_{M/p_i^2}[x_j]\mathbb{B}_{M/p_i^2}[b] > 0$. By Lemma 7.1, this implies that

$$A_{x_j,b} \cap (\ell_k(x_j) \cup \ell_k(x_k)) = \emptyset.$$

Hence $A_{x_j,b} \subset \ell_i(x_j)$, and the first conclusion of the lemma follows.

Finally, we prove (9.36). Suppose that $A_{x_k,b_k} \subset \ell_i(x_k)$. Then

$$\phi(p_i^2) = \mathbb{A}_{M/p_i^2}[x_k]\mathbb{B}_{M/p_i^2}[b_k] = \mathbb{A}_{M/p_i^2}[x_j]\mathbb{B}_{M/p_i^2}[b_k],$$

hence $A_{x_j,b_k} \subset \ell_i(x_j)$ as claimed. The same argument works in the other direction. □

Lemma 9.10. *Assume that (DB2) holds, and let $b \in B$. Then:*

(i) *if $A_{x_j,b} \subset \ell_i(x_j)$, then*

$$\mathbb{A}_{M/p_i^2}[x_j]\mathbb{B}_{M/p_i^2}[b] = \phi(p_i^2), \tag{9.39}$$

(ii) *if $A_{x_j,b} \subset \ell_k(x_j)$, then*

$$\mathbb{A}_{M/p_k^2}[x_j] = p_k,\ \mathbb{B}_{M/p_k^2}[b_k] = \phi(p_k), \tag{9.40}$$

(iii) *if $A_{x_j,b} \subset \ell_k(x_k)$, then*

$$\mathbb{B}_{M/p_ip_k}[b] = 1. \tag{9.41}$$

The same statements hold with j and k interchanged.

Proof. The first statement is true since $x_k \notin A$. For (ii), since $M/p_k \in \mathrm{Div}(A)$, we must have $\mathbb{A}_{M/p_k^2}[x_j]\mathbb{B}_{M/p_k^2}[b] = \phi(p_k^2)$, implying (9.40).

We now prove (iii). The assumption that $A_{x_j,b} \subset \ell_k(x_k)$ implies

$$1 = \frac{1}{\phi(p_ip_k)}\mathbb{A}_{M/p_ip_k}[x_j|\ell_k(x_k)]\mathbb{B}_{M/p_ip_k}[b] + \frac{1}{\phi(p_ip_k^2)}\mathbb{A}_{M/p_ip_k^2}[x_j|\ell_k(x_k)]\mathbb{B}_{M/p_ip_k^2}[b].$$

On the other hand, since $p_i = 2$, we have by (9.34)

$$\mathbb{A}_{M/p_ip_k}[x_j|\ell_k(x_k)] = \mathbb{A}_{M/p_k}[x_k] = \phi(p_k) = \phi(p_ip_k).$$

If $\mathbb{B}_{M/p_ip_k}[b] > 0$, this implies (9.41) and we are done. Suppose now that $\mathbb{B}_{M/p_ip_k}[b] = 0$. Then

$$\mathbb{A}_{M/p_ip_k^2}[x_j|\ell_k(x_k)]\mathbb{B}_{M/p_ip_k^2}[b] = \phi(p_ip_k^2) = \phi(p_k^2),$$

and by (9.34) we have $M/p_k, M/p_k^2 \in \mathrm{Div}(A)$. Hence $\mathbb{B}_{M/p_ip_k^2}[b] = 1$, so that

$$\mathbb{A}_{M/p_ip_k^2}[x_j|\ell_k(x_k)] = \mathbb{A}_{M/p_k^2}[x_k] = \phi(p_k^2).$$

But now

$$|A \cap \Pi(x_k, p_j^2)| \geq \mathbb{A}_{M/p_k^2}[x_k] + \mathbb{A}_{M/p_k}[x_k] = \phi(p_k^2) + \phi(p_k) = p_k^2 - 1 > p_ip_k,$$

contradicting Lemma 7.3. □

Lemma 9.11. *Assume that (DB2) holds, and that there exist $b_i, b_k \in B$ such that $A_{x_j,b_i} \subset \ell_i(x_j)$ and $A_{x_j,b_k} \subset \ell_k(x_j)$. Then we have the following:*

(i) $\mathbb{A}_{M/p_i^2}[x_j] = 1$ *and* $\mathbb{B}_{M/p_i^2}[b] = p_i$ *(in particular, $M/p_i \in \mathrm{Div}(B)$),*
(ii) $\Phi_{p_j^2}|A$, $A \subset \Pi(x_j, p_j)$, *and* $A_{x_k} \cap \ell_j(x_k) = \emptyset$,
(iii) $A_{x_k,b_k} \subset \ell_j(x_j)$, *with* $\mathbb{B}_{M/p_ip_j}[b_k] = 1$.

The same statement holds with j and k interchanged.

Proof. Let b_i, b_k be as in the assumptions of the lemma. Then b_i satisfies (9.39), and b_k satisfies (9.40). Hence

$$\begin{aligned} |A \cap \Pi(x_j, p_j^2)| &\geq \mathbb{A}_{M/p_k}[x_k] + \mathbb{A}_{M/p_k^2}[x_j] + \mathbb{A}_{M/p_i^2}[x_j] \\ &= \phi(p_k) + p_k + \mathbb{A}_{M/p_i^2}[x_j] \\ &= p_ip_k - 1 + \mathbb{A}_{M/p_i^2}[x_j]. \end{aligned}$$

By Lemma 7.3, the last line must be less than or equal to p_ip_k. Hence $\mathbb{A}_{M/p_i^2}[x_j] = 1$, and by (9.39) we must have $\mathbb{B}_{M/p_i^2}[b_i] = \phi(p_i^2) = 2$, proving (i). Furthermore, it follows that

$$|A \cap \Pi(x_j, p_j^2)| = p_ip_k.$$

Hence

$$|A \cap \Pi(x_j, p_j)| \geq |A \cap \Pi(x_j, p_j^2)| + \mathbb{A}_{M/p_j}[x_j] > p_ip_k.$$

By Corollary 7.4, we have $\Phi_{p_j^2}|A$ and $A \subset \Pi(x_j, p_j)$. In particular, $M/p_j^2 \notin \mathrm{Div}(x_\nu, A)$ for $\nu \in \{j, k\}$. Since we also have $(x_k * F_j) \cap A = \emptyset$, it follows that $A_{x_k} \cap \ell_j(x_k) = \emptyset$. This proves (ii).

We now prove (iii). By Lemma 9.9 with j and k interchanged, A_{x_k,b_k} must be contained in one of the lines $\ell_i(x_k), \ell_j(x_j), \ell_j(x_k)$.

- Suppose that $A_{x_k,b_k} \subset \ell_i(x_k)$. By (9.36), this would imply $A_{x_j,b_k} \subset \ell_i(x_j)$, contradicting the choice of b_k.
- Suppose now that $A_{x_k,b_k} \subset \ell_j(x_k)$. Then (9.40) holds with j and k interchanged, and in particular $\mathbb{A}_{M/p_j^2}[x_k] > 0$, contradicting part (ii) of the lemma.

Hence $A_{x_k,b_k} \subset \ell_j(x_k)$, and the last part follows from (9.41) with j and k interchanged. □

Lemma 9.12. *Assume (DB2). For every $b \in B$, we have the following:*

(i) $\mathbb{B}_{M/p_ip_j}[b]\mathbb{B}_{M/p_ip_k}[b] = 0$,
(ii) *if $A_{x_j,b} \subset \ell_k(x_k)$, then $A_{x_k,b} \subset \ell_j(x_k)$.*

The same holds with j and k interchanged.

Proof. Assume, by contradiction, that $\mathbb{B}_{M/p_ip_j}[b]\mathbb{B}_{M/p_ip_k}[b] > 0$ for some $b \in B$. Let $y \in \mathbb{Z}_M$ be the unique point with $(b - y, M) = M/p_i$. Then there exist $b_j, b_k \in B$ such that $(y - b_\nu, M) = M/p_\nu$ for $\nu = j, k$.

Let $a \in A$, and consider the saturating set $B_{y,a}$. Then $B_{y,a}$ is contained in the vertices of the M-cuboid with vertices at b, b_j, b_k and y. By (DB2), we have

$$\{M/p_j, M/p_k, M/p_ip_jp_k\} \subset \mathrm{Div}(A), \tag{9.42}$$

hence no other elements of B are permitted at the vertices of that cuboid except possibly at the point u with $(u - b, M) = M/p_jp_k$. We consider two cases.

Case 1. Suppose that $u \notin B$. By Lemma 7.1, $B_{y,a}$ must consist of just one of the points b, b_j, b_k. The latter, in turn, implies that a must be contained in an M-fiber in some direction, which is clearly false for any $a \in A \cap \Lambda$.

Case 2. Assume now that $u \in B$. In this case, we will use a splitting argument. By translational invariance, we may assume that $x_j = b = 0$, so that $\Lambda = \Lambda(0, D(M))$, $x_k = M/p_i$, and $A_1 = (F_j \setminus \{0\}) \cup ((x_k * F_k) \setminus \{x_k\})$. Replacing B by rB for some $r \in R$ if necessary, we may assume that

$$b_j = M/p_i + M/p_j,\ b_k = M/p_i + M/p_k,\ u = M/p_j + M/p_k.$$

By (9.42), no other points of Λ are permitted in B, so that

$$B \cap \Lambda = \{0, b_j, b_k, u\}. \tag{9.43}$$

Let $z = \mu M/p_j + \nu M/p_k$ for some $\mu \in \{2, 3, \ldots, p_j - 1\}$ and $\nu \in \{2, 3, \ldots, p_k - 1\}$, and let $a' \in A, b' \in B$ be the elements such that $a' + b' = z$. We first consider splitting for the fiber $z * F_j$. We have

$$\nu M/p_k = b_k + (M/p_i + (\nu - 1)M/p_k),\ \text{with } b_k \in B,\ M/p_i + (\nu - 1)M/p_k \in A,$$
$$\nu M/p_k + M/p_j = b_j + (M/p_i + \nu M/p_k),\ \text{with } b_j \in B,\ M/p_i + \nu M/p_k \in A.$$

Hence $z * F_j$ splits with parity (A, B), so that

$$a' \in \Pi(0, p_j^2),\ b' \in \Pi(z, p_j^2). \tag{9.44}$$

Next, consider the fiber $z * F_k$. We have

$$\mu M/p_j = 0 + \mu M/p_j,\quad \text{with } 0 \in B,\ \mu M/p_j \in A,$$
$$\mu M/p_j + M/p_k = u + (\mu - 1)M/p_j,\quad \text{with } u \in B,\ (\mu - 1)M/p_j \in A.$$

Hence $z * F_k$ also splits with parity (A, B), so that

$$a' \in \Pi(0, p_k^2),\ b' \in \Pi(z, p_k^2). \tag{9.45}$$

By (9.44) and (9.45), we have

$$a' \in \ell_i(0),\ b' \in \ell_i(z).$$

This together with (9.43) implies that

$$\mathbb{B}_{M/p_i^2}[\mu M/p_j + \nu M/p_k] \geq 1 \ \ \forall \mu \in \{2, 3, \dots, p_j - 1\}, \nu \in \{2, 3, \dots, p_k - 1\}. \tag{9.46}$$

However, since $p_i = 2$, by (9.42) we also have $\mathbb{B}_{M/p_j}[b'] = \mathbb{B}_{M/p_k}[b'] = 0$ and $\mathbb{B}_{M/p_ip_j}[b']$, $\mathbb{B}_{M/p_ip_k}[b'] \leq 1$. At least one of p_j and p_k is greater than or equal to 5, say $p_k \geq 5$. Applying (9.46) with $\mu = 2$ and $\nu = 2, 3, 4$, we get a contradiction. This completes the proof of (i).

We now prove (ii). Assume that $A_{x_j,b} \subset \ell_k(x_k)$ for some $b \in B$. By Lemma 9.10, we have $\mathbb{B}_{M/p_ip_k}[b] = 1$. Consider now $A_{x_k,b}$.

- If $A_{x_k,b} \subset \ell_j(x_j)$, then we also have $\mathbb{B}_{M/p_ip_j}[b] = 1$, contradicting (i).
- We cannot have $A_{x_k,b} \subset \ell_i(x_k)$, since that would contradict (9.36).

It follows by Lemma 9.9 that $A_{x_k,b} \subset \ell_j(x_k)$, as claimed. □

Proposition 9.13. *Assume (DB2). Then* $\mathbb{B}_{M/p_ip_j}[b] = \mathbb{B}_{M/p_ip_k}[b] = 0$ *for all* $b \in B$.

Proof. Assume for contradiction that $(b - b_k, M) = M/p_ip_k$. We again use a splitting argument. By translational invariance, we may assume that $x_j = b = 0$, so that $\Lambda = \Lambda(0, D(M))$, $x_k = M/p_i$, and $A_1 = (F_j \setminus \{0\}) \cup ((x_k * F_k) \setminus \{x_k\})$. Replacing B by rB for some $r \in R$ if necessary, we may assume that $b' = M/p_i + M/p_k$.

Let $z = \mu M/p_j + \nu M/p_k$ for some $\mu \in \{1, 2, \dots, p_j - 1\}$ and $\nu \in \{2, 3, \dots, p_k - 1\}$, and let $a_z \in A, b_z \in B$ be the elements such that $a_z + b_z = z$. We first consider splitting for the fiber $z * F_j$. Since

$$\nu M/p_k = b' + (M/p_i + (\nu - 1)M/p_k),\ \text{with } b' \in B,\ M/p_i + (\nu - 1)M/p_k \in A,$$

we have

$$\{a_z, b_z\} \subset \Pi(0, p_j^2) \cup \Pi(z, p_j^2). \tag{9.47}$$

Next, consider the fiber $z * F_k$. Since

$$\mu M/p_j = 0 + \mu M/p_j, \quad \text{with } 0 \in B,\ \mu M/p_j \in A,$$

it follows that

$$\{a_z, b_z\} \subset \Pi(0, p_k^2) \cup \Pi(z, p_k^2). \tag{9.48}$$

Combining (9.47) and (9.48), we have

$$\{a_z, b_z\} \subset \ell_i(0) \cup \ell_i(z) \cup \ell_i(\mu M/p_j) \cup \ell_i(\nu M/p_k).$$

Taking also into account that $a_z + b_z = z$, we only need to consider the following two cases:

$$\{a_z, b_z\} \subset \ell_i(0) \cup \ell_i(z), \tag{9.49}$$

$$\{a_z, b_z\} \subset \ell_i(\mu M/p_j) \cup \ell_i(\nu M/p_k). \tag{9.50}$$

Case 1. Assume that (9.49) holds. We cannot have $a_z, b_z \in \Lambda$, since $A \cap \Lambda$ has no points on these two lines. Hence $(a_z, p_i) = (b_z, p_i) = 1$.

Suppose that $a_z \in \ell_i(z)$ and $b_z \in \ell_i(0)$. Then

$$(a_z - \mu M/p_j, M) = M/p_i^2 p_k = (b_z - b', M),$$

contradicting divisor exclusion. Hence $a_z \in \ell_i(0)$ and $b_z \in \ell_i(z)$, with

$$(a_z, M) = (b_z - z, M) = M/p_i^2. \tag{9.51}$$

Note that this implies that

$$(a_z - (M/p_i + M/p_k), M) = M/p_i^2 p_k \in \mathrm{Div}(A). \tag{9.52}$$

Case 2. Assume that (9.50) holds. Suppose first that $a_z, b_z \in \Lambda$, then $a_z \in \{\mu M/p_j, M/p_i + \nu M/p_k\}$. If $a_z = \mu M/p_j$, then $b_z - b = b_z = \nu M/p_k$, contradicting divisor exclusion since $M/p_k \in \mathrm{Div}(A)$. If on the other hand $a_z = M/p_i + \nu M/p_k$, then $b_z = M/p_i + \mu M/p_j$, contradicting Lemma 9.12 (i).

Hence $a_z, b_z \notin \Lambda$. If $a_z \in \ell_i(\nu M/p_k)$ and $b_z \in \ell_i(\mu M/p_j)$, it follows that

$$(a_z - \mu M/p_j, M) = M/p_i^2 p_j p_k = (b_z - b', M),$$

contradicting divisor exclusion. Hence $a_z \in \ell_i(\mu M/p_j)$ and $b_z \in \ell_i(\nu M/p_k)$, with

$$(a_z - \mu M/p_j, M) = (b_z - \nu M/p_k, M) = M/p_i^2.$$

In this case, we have

$$(b' - b_z, M) = M/p_i^2 p_k \in \mathrm{Div}(B). \tag{9.53}$$

We now allow μ and ν to vary. Clearly, (9.52) and (9.53) are mutually exclusive, so that we are either always in Case 1 or always in Case 2. More precisely, either (9.51) holds for all μ, ν, so that

$$\mathbb{B}_{M/p_i^2}[\mu M/p_j + \nu M/p_k] \geq 1 \ \ \forall \mu \in \{1, 2, \dots, p_j - 1\}, \nu \in \{2, 3, \dots, p_k - 1\}, \tag{9.54}$$

or else (9.52) holds for all μ, ν, so that

$$\begin{aligned} &\mathbb{A}_{M/p_i^2}[\mu M/p_j] \geq 1 \ \ \forall \mu \in \{1, 2, \dots, p_j - 1\}, \\ &\mathbb{B}_{M/p_i^2}[\nu M/p_k] \geq 1 \ \ \forall \nu \in \{2, 3, \dots, p_k - 1\}. \end{aligned} \tag{9.55}$$

Since $p_i = 2$, by (9.42) we have $\mathbb{B}_{M/p_j}[b''] = \mathbb{B}_{M/p_k}[b''] = 0$ and $\mathbb{B}_{M/p_i p_j}[b''], \mathbb{B}_{M/p_i p_k}[b''] \leq 1$ for all $b'' \in B$. As in the proof of Lemma 9.12 (i), this means that (9.54) cannot hold. It remains to consider the case when (9.55) holds. In this case, we have

$$\mathbb{A}_{M/p_i p_k}[x_j]\mathbb{B}_{M/p_i p_k}[0] = p_k - 1 = \phi(p_i p_k),$$

so that $A_{x_j,0} \subset \ell_k(x_k)$. By Lemma 9.12 (ii) with j and k interchanged, we have $A_{x_k,0} \subset \ell_j(x_k)$, so that by (9.40) with j and k interchanged,

$$\mathbb{A}_{M/p_j^2}[x_k] = p_j.$$

Hence

$$\begin{aligned} |A \cap \Pi(0, p_k^2)| &\geq \mathbb{A}_{M/p_j}[x_j] + \mathbb{A}_{M/p_j^2}[x_k] + \sum_{\mu=1}^{p_j-1} \mathbb{A}_{M/p_i^2}[\mu M/p_j] \\ &\geq (p_j - 1) + p_j + (p_j - 1) \\ &= p_i p_j + (p_j - 2) > p_i p_j, \end{aligned}$$

contradicting Lemma 7.3.

This proves that $M/p_ip_k \notin \mathrm{Div}(B)$. The proof that $M/p_ip_j \notin \mathrm{Div}(B)$ is identical, with the j and k indices interchanged. This ends the proof of the proposition. $\square$

We are now ready to complete the proof of Theorem 9.1 under the assumption (DB2). We first prove (9.35). If we had $A_{x_j,b} \subset \ell_k(x_k)$ or $A_{x_k,b} \subset \ell_j(x_j)$ for some $b \in B$, it would follow by Lemma (9.10) (iii) that $\{M/p_ip_j, M/p_ip_k\} \cap \mathrm{Div}(B) \neq \emptyset$; however, that is impossible by Proposition 9.13. By Lemmas 9.9 and 9.10, for every $b \in B$ we must have either

$$A_{x_j,b} \subset \ell_i(x_j) \tag{9.56}$$

or

$$A_{x_j,b} \subset \ell_k(x_j). \tag{9.57}$$

Furthermore, if there were elements $b_i, b_k \in B$ such that (9.56) holds for b_i and (9.57) holds for b_k, it would follow by Lemma 9.11 (iii) that $\mathbb{B}_{M/p_ip_j}[b_k] = 1$, which, again, is impossible by Proposition 9.13. The same holds with the indices i and j interchanged.

Hence either (9.56) holds for all $b \in B$, or (9.57) holds for all $b \in B$. In the first case, we have $B = B_0$ by (9.36). In the second case, also by (9.36), we have $B = B_1$.

Assume first that $B = B_1$. Then $A_{x_j} \subset \ell_k(x_j)$. By (9.40), it follows that (A, B) has a (1,2)-cofibered structure in the p_k direction, with a cofiber in A at distance M/p_k^2 from x_j. We now use Lemma 3.11 to shift that cofiber to x_j. We then do the same with j and k indices interchanged, using that $A_{x_k} \subset \ell_j(x_k)$. Let A' be the set thus obtained, so that $A' \cap \Lambda = \{x_j, x_k\} * (F_j \cup F_k)$. Then A' is T2-equivalent to A, and satisfies (DB3). By Proposition 9.2, A' is T2-equivalent to Λ, therefore so is A.

It remains to consider the case when $B = B_0$. By (9.39), for each $b \in B$ we have either

$$\mathbb{A}_{M/p_i^2}[x_j] = 1, \mathbb{B}_{M/p_i^2}[b] = p_i, \text{ and } M/p_i \in \mathrm{Div}(B), \tag{9.58}$$

or

$$\mathbb{A}_{M/p_i^2}[x_j] = p_i, \mathbb{B}_{M/p_i^2}[b] = 1, \text{ and } M/p_i \in \mathrm{Div}(A). \tag{9.59}$$

We claim that

$$\text{(9.59) holds for all } b \in B. \tag{9.60}$$

Assume, by contradiction, that (9.58) holds for some $b_0 \in B$. Then $\mathbb{A}_{M/p_i^2}[x_j] = 1$, hence we cannot have (9.59) for any $b \in B$. Therefore, all $b \in B$ must satisfy (9.58). But this implies that

$$\mathbb{B}_M[b] + \mathbb{B}_{M/p_i}[b] + \mathbb{B}_{M/p_i^2}[b] = p_i^2 \quad \forall b \in B,$$

and in particular p_i^2 must divide $|B|$, which is not allowed. This proves the claim.

We now see from (9.60) that (A, B) has a (1,2)-cofibered structure in the p_i direction, with a cofiber in A at distance M/p_i^2 from x_j. We now Lemma 3.11 to shift that cofiber to x_j. Let A' be the set thus obtained, so that $x_j * F_i \in A'$ and $A \cap (\Lambda \setminus (x_j * F_i)) = A' \cap (\Lambda \setminus (x_j * F_i))$. Then A' is T2-equivalent to A, and contains a p_i corner structure. By Theorem 8.3, A' (therefore A) is T2-equivalent to B. This completes the proof of the theorem.

10. Fibered grids

10.1. **Main results for fibered grids.** Throughout most of this section we will work under the following assumption.

Assumption (F): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M | A$, and A is fibered on $D(M)$-grids.

We are not making any assumptions on the parity of M at this point, but some of the arguments below will differ between the odd and even cases. We will indicate this as appropriate.

Let $\mathcal{I}$ be the set of elements of A that belong to an M-fiber in the p_i direction, that is,

$$\mathcal{I} = \{a \in A : \ \mathbb{A}_{M/p_i}[a] = \phi(p_i)\}.$$

The sets $\mathcal{J}$ and $\mathcal{K}$ are defined similarly. The assumption (F) implies that every element of A belongs to an M-fiber in some direction, hence $A = \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$. We emphasize that this does *not* have to be a disjoint union and that it is possible for an element of A to belong to two or three of these sets.

Our main result on fibered grids is the following theorem.

Theorem 10.1. *Assume that (F) holds.*

(I) *If $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} \neq \emptyset$, then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$, where $\Lambda := \Lambda(0, D(M))$. By Corollary 2.2, both A and B satisfy (T2).*

(II) *Assume that $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset$. Then, after a permutation of the i, j, k indices if necessary, the following holds.*

- (II a) *At least one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty. Without loss of the generality, we may assume that $\mathcal{I} = \emptyset$, so that $A \subset \mathcal{J} \cup \mathcal{K}$.*
- (II b) *If $A \subset \mathcal{J}$ or $A \subset \mathcal{K}$, then A is M-fibered in the p_j or p_k direction, respectively. Consequently, the conditions of Theorem 5.2 are satisfied in that direction. By Corollary 5.3, both A and B satisfy (T2).*
- (II c) *Suppose that $\mathcal{I} = \emptyset$, and that $\mathcal{J} \setminus \mathcal{K}$ and $\mathcal{K} \setminus \mathcal{J}$ are both nonempty.*
  - *If $\Phi_{p_i} | A$, then, after interchanging A and B, the conditions of Theorem 5.2 are satisfied in the p_i direction. By Corollary 5.3, both A and B satisfy (T2).*
  - *If $\Phi_{p_i^2} | A$, then $A \subset \Pi(a, p_i)$ for any $a \in A$. By Theorem 5.1, both A and B satisfy (T2).*

Theorem 10.1 extends [25, Theorem 9.1], covering both odd and even cases. Part (I) of the theorem was already proved in [25, Corollary 9.6], with no assumptions on the parity of M. We will therefore assume from now on that

$$\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset. \tag{10.1}$$

Moreover, part (II b) is simply an application of Theorem 5.2 and Corollary 5.3 to a fibered set. Again, no parity assumptions are needed.

It remains to prove (II a) and (II c). We start with preliminaries in Section 10.2. In Section 10.3, we consider the fibering of $\mathcal{I}, \mathcal{J}, \mathcal{K}$ on lower scales; in particular, we investigate the possibility that they might have unfibered grids on scale N_ν if $\Phi_{N_\nu}|A$ for some $\nu \in \{i,j,k\}$. Our results in that regard both strengthen those of [25] and extend them to the even case.

In Section 11, we discuss splitting for fibered grids. Our main structure result is Lemma 11.2, which can be thought of as an easier "local" version of Theorem 10.1 (II a) restricted to individual $D(M)$-grids. We also investigate questions related to localization and splitting parity. The results of this section are not restricted to $n_i = n_j = n_k = 2$, and provide partial structural information in the more general case. Based on this work, in Section 12.1 we identify two relatively easy cases when Theorem 10.1 (II a) has a short proof.

We prove part (II a) of Theorem 10.1 in Section 12. In [25], the proof of (II a) in the odd case is organized according to whether the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are pairwise disjoint or not. We depart from that here, considering instead the following sets of assumptions.

Assumption (F1): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, (10.1) holds, and $\Phi_{M_\nu}|A$ for some $\nu \in \{i,j,k\}$.

Assumption (F2): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, (10.1) holds, and $\Phi_{M_\nu} \nmid A$ for all $\nu \in \{i,j,k\}$.

It turns out that these sets of assumptions are better suited to the problem at hand, in the sense that both (F1) and its complementary assumption (F2) provide directly actionable information. If $\Phi_{M_\nu}|A$ for some $\nu \in \{i,j,k\}$, say $\nu = k$, then $\mathcal{K}$ mod M_k is a union of M_k-fibers in the other two directions. Given the condition that $|A| = p_i p_j p_k$, such lower-scale fibers saturate the plane bound in Lemma 7.3. This leads to strict constraints on the distribution of the fibers in different directions in A. We then prove that these constraints cannot be met when fibers in all three directions are present. If, on the other hand, $\Phi_{M_\nu}|B$ for all $\nu \in \{i,j,k\}$, then this leads to very strong fibering properties of B that are not compatible with $\mathcal{I}, \mathcal{J}, \mathcal{K}$ all being nonempty.

We note that the proof of Theorem 10.1 (II a) under the assumption (F2) is much shorter than for (F1). Working backwards from the conclusion, once we know that one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty, we can conclude from Lemma 9.8 in [25] (see Lemma 10.3 below) that (F2) can never actually hold. This might provide an *a posteriori* explanation of why the case (F1) is more difficult.

Finally, we prove Theorem 10.1 (II c) in Section 13. With parts (II a) and (II b) of the theorem in place, we are left with the following assumption.

Assumption (F3): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, (10.1) holds, $\mathcal{I} = \emptyset$, $\mathcal{J} \setminus \mathcal{K} \neq \emptyset$, and $\mathcal{K} \setminus \mathcal{J} \neq \emptyset$.

We need to prove that, assuming (F3), we can apply either the slab reduction or the subgroup reduction as indicated in the theorem. The proof follows the same general lines as the corresponding argument in [25], but with the even case included. We are also able to simplify the argument in several places by using Lemma 5.6 and Corollary 5.7 instead of verifying the conditions of Theorem 5.2 directly. (Compare e.g. our proof of Corollary 13.7 to the proof of Lemmas 9.37 and 9.38 in [25].)

10.2. **Toolbox for fibered grids.** We will continue to use the notation of [25]. Given $N|M$, we will use $\mathbb{I}^N$, $\mathbb{J}^N$, $\mathbb{K}^N$ to denote the N-boxes associated with $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ respectively. Recall also that $M_\nu = M/p_\nu^2$ for $\nu \in \{i,j,k\}$. We will need several lemmas from [25]. Lemmas 10.2–10.4 do not require any assumptions on the parity of M, so that the same results hold for any permutation of the indices i,j,k.

Lemma 10.2. [25, Lemma 9.5]. *Assume (F) and (10.1), and let $a \in \mathcal{J} \cap \mathcal{K}$. Then $A \cap \Pi(a, p_i^2) = a * F_j * F_k$.*

Proof. If $A \cap \Lambda(a, D(M))$ were M-fibered in the p_i direction, we would have $a \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, contradicting (10.1). Hence $A \cap \Lambda(a, D(M))$ is M-fibered in at least one of the other two directions, say p_j. Then every point of $a * F_k$ belongs to an M-fiber in the p_j direction in A, proving that $a * F_j * F_k \subset A \cap \Pi(a, p_i^2)$. The equality follows from Lemma 7.3. □

Lemma 10.3. [25, Lemma 9.8] *Assume (F). For each $\alpha \in \{1,2\}$, we have*

$$\Phi_{M/p_k^{\alpha_k}} | A \;\Leftrightarrow\; \Phi_{M/p_k^{\alpha_k}} | \mathcal{K}.$$

In particular, if $\Phi_{M/p_k^{\alpha_k}} \nmid A$ for some $\alpha_k \in \{1,2\}$, then $\mathcal{K} \neq \emptyset$.

Lemma 10.4. [25, Lemma 9.9] *Let $A \oplus B = \mathbb{Z}_M, M = p_i^2 p_j^2 p_k^2, |A| = |B| = p_i p_j p_k$. Then*

$$\{m/p_k^\alpha : \; \alpha \in \{0,1,2\}, m \in \{M, M/p_i, M/p_j, M/p_i p_j\}\} \cap \mathrm{Div}(B) \neq \emptyset.$$

10.3. **Fibering on lower scales.** Our first lemma here is [25, Lemma 9.11]. While it was only stated there for odd M, the parity assumption is not needed in the proof.

Lemma 10.5. *Assume that (F) holds.*

(i) Let $\Lambda := \Lambda(a_0, D(N_i))$ for some $a_0 \in \mathcal{I}$. If $\mathcal{I} \cap \mathcal{J} \cap \Lambda = \emptyset$ and A is N_i-fibered on Λ, it cannot be fibered in the p_j direction.

(ii) If $\Phi_{N_i} | B$ and

$$\{M/p_j, M/p_k, M/p_i p_j, M/p_i p_k\} \cap \mathrm{Div}(B) = \emptyset, \tag{10.2}$$

then B must be N_i-fibered in the p_i direction. Note in particular that if $\{D(M)|m|M\} \subset \mathrm{Div}(A)$, then (10.2) holds.

By Lemma 10.3, we have $\Phi_{N_k}|A$ if and only if $\Phi_{N_k}|\mathcal{K}$. We may then consider $\mathcal{K}$ mod N_k as a multiset in $\mathbb{Z}_{N_k}$ with constant multiplicity p_k, and ask whether it might have unfibered grids on that scale. The remaining results in this section strengthen [25, Lemma 9.12 and Corollary 9.13] in two ways: we allow M to be even, and we prove that the corner structure (permitted in the proof of [25, Lemma 9.12]) cannot actually occur.

Lemma 10.6. *Assume (F), and suppose that $\Phi_{N_k}|A$. Then $\mathcal{K}$ mod N_k cannot have an extended corner structure (see Definition 8.1) on any $D(N_k)$-grid Λ.*

Proof. Assume for contradiction that $\mathcal{K}$ does contain a corner structure. Then, possibly after interchanging i and j, we must have one of the two following cases.

Case 1 (p_k corner): There exist $a, a' \in \mathcal{K}$ with $(a - a', M) = M/p_k^2$ such that

$$\mathbb{A}^{N_k}_{N_k/p_i}[a] = p_k \phi(p_i) \text{ and } \mathbb{A}^{N_k}_{N_k/p_j}[a'] = p_k \phi(p_j).$$

However, this means

$$|A\cap\Pi(a,p_j^2)|\geq \mathbb{A}^{N_k}_{N_k}[a]+\mathbb{A}^{N_k}_{N_k}[a']+\mathbb{A}^{N_k}_{N_k/p_i}[a]$$
$$=p_k(p_i+1)>p_ip_k$$

contradicting Lemma 7.3.

Case 2 (p_i corner): There exist $a,a'\in\mathcal{K}$ with $(a-a',M)=M/p_i$ such that

$$\mathbb{A}^{N_k}_{N_k/p_k}[a]=\phi(p_k^2) \text{ and } \mathbb{A}^{N_k}_{N_k/p_j}[a']=p_k\phi(p_j). \tag{10.3}$$

We notice that

(i) $\{M/p_k,M/p_k^2\}\subset \mathrm{Div}(\ell_k(a))\subset \mathrm{Div}(A)$.
(ii) $\{D(M)|m|M\}\subset \mathrm{Div}(a,A\cap\Pi(a',p_i^2))\subset\mathrm{Div}(A)$, hence (7.4) holds.

Suppose first that $a*F_j*F_k\subset A$. Then $A\cap\Pi(a,p_i^2)$ contains p_jp_k points of $a*F_j*F_k$, as well as the additional $\phi(p_k^2)$ points of $\mathcal{K}$ at distance M/p_k^2 from a, contradicting Lemma 7.3.

We must therefore have $a*F_j*F_k\not\subset A$. Let $x\in(a*F_j*F_k)\setminus A$. By (10.3) and (ii) above, x satisfies the conditions of Lemma 7.5, with the flat corner configuration perpendicular to the p_k direction. However, the conclusion of Lemma 7.5 implies that $M/p_k^2\in\mathrm{Div}(B)$, contradicting (i). □

Lemma 10.7. *Assume that (F) holds with odd M. Suppose that $\Phi_{N_k}|A$ and that there exists a $D(N_k)$-grid Λ on which $\mathcal{K}$ is not N_k-fibered. Then $\mathcal{K}\cap\Lambda$ mod N_k has either the p_k full plane structure, or a p_i or p_j almost corner structure (see Lemma 8.7 (i) and (iii) for the definitions). Moreover:*

(i) $\{D(M)|m|M\}\cup\{M/p_k^2,M/p_ip_jp_k^2\}\subset\mathrm{Div}(A)$,
(ii) *there exists an $x\in\mathbb{Z}_M\setminus\mathcal{K}$ such that $\mathbb{K}^{N_k}_{N_k/p_k}[x]=\phi(p_k^2)$.*

Proof. Assume that $\Phi_{N_k}|A$, hence $\Phi_{N_k}|\mathcal{K}$ as noted above. Assume also that $\mathcal{K}$ is not fibered on a $D(N_k)$-grid Λ. We claim that

$$\{m:\ D(N_k)|m|N_k\}\not\subset\mathrm{Div}_{N_k}(\mathcal{K}); \tag{10.4}$$

moreover, this holds without any assumptions on the parity of M. Indeed, suppose that (10.4) fails. This means that for every $D(N_k)|m|N_k$ there exist $a,a'\in\mathcal{K}$ (depending on m) such that $(a-a',N_k)=m$. Since $\mathbb{K}^{N_k}_{N_k}[a]=\mathbb{K}^{N_k}_{N_k}[a']=\phi(p_k)$, we have

$$\big\{m/p_k^\alpha:\ \alpha\in\{0,1,2\},m\in\{M,M/p_i,M/p_j,M/p_ip_j\}\big\}\subset\mathrm{Div}(\mathcal{K}).$$

This contradicts Lemma 10.4.

We now go back to our initial assumption that M (therefore N_k) is odd. By the assumptions of the lemma together with (10.4), we may apply Lemma 8.7 with $N=N_k$ to $\mathcal{K}$ mod N_k on Λ. Therefore, one of the conclusions (i)–(iii) of Lemma 8.7 must hold. Since a corner structure as in (ii) is prohibited by Lemma 10.6, we must be in one of the cases (i) or (iii).

An additional constraint is that any plane $\Pi(x,p_i^2)$ in $\mathbb{Z}_{N_k}$ may contain at most p_j distinct points of $\mathcal{K}$ mod N_k, each of multiplicity p_k. Indeed, if there were more than p_j such points, this would mean that $\mathcal{K}$ has more than p_jp_k points in the plane $\Pi(x,p_i^2)$ in $\mathbb{Z}_M$, contradicting Lemma 7.3. The same holds with i and j interchanged.

Among the structures in Lemma 8.7 (i) and (iii) with $N=N_k$, the only ones that meet this constraint are as follows:

- $\mathcal{K}\cap\Lambda$ has the p_k full plane structure as in Lemma 8.7 (i),
- $\mathcal{K}\cap\Lambda$ has a p_i or p_j almost corner structure as in Lemma 8.7 (iii).

In both cases, possibly after a permutation of i and j, there exist $x_1,x_2,x_3\in\mathbb{Z}_M$ such that $(x_\nu-x_{\nu'},N_k)=N_k/p_i$ for $\nu\neq\nu'$, and

$$\mathbb{K}^{N_k}_{N_k/p_k}[x_1]=\mathbb{K}_{M/p_k^2}[x_1]=\phi(p_k^2), \tag{10.5}$$

and

$$\mathbb{K}^{N_k}_{N_k/p_j}[x_2]=\mathbb{K}^{N_k}_{N_k/p_j}[x_3]=p_k\phi(p_j).$$

Now consider $\mathcal{K}$ on scale M. By (10.5), (ii) holds with $x=x_1$. Furthermore, $\mathcal{K}_0:=\{x_2,x_3\}*(F_j\setminus\{0\})*F_k\subset\mathcal{K}$. Hence

$$\begin{aligned}&\{D(M)|m|M\}\subset \mathrm{Div}(\mathcal{K}_0)\subset\mathrm{Div}(A),\\ &M/p_k^2\in\mathrm{Div}(A\cap\ell_k(x_1))\subset\mathrm{Div}(A),\\ &M/p_ip_jp_k^2\in\mathrm{Div}(A\cap\ell_k(x_1),\mathcal{K}_0)\subset\mathrm{Div}(A),\end{aligned}$$

proving (i). □

Corollary 10.8. *Assume that (F) holds and that $p_k>\min_\nu p_\nu$. Then:*

(i) $\mathcal{K}$ cannot be N_k-fibered on any $D(N_k)$-grid in the p_k direction.

(ii) If $\Phi_{N_k}|A$, then $\mathcal{K}$ is N_k-fibered on each $D(N_k)$-grid in one of the p_i and p_j directions. In particular, $\mathcal{K}\subset\mathcal{I}\cup\mathcal{J}$.

Proof. Part (i) is a simple consequence of Lemma 7.3. The proof of (ii) for odd M appears in [25], Corollary 9.13.

We prove (ii) under the assumption $p_i=2$. Since $\Phi_{N_k}|A$, by Lemma 10.3 we have $\Phi_{N_k}|\mathcal{K}$. Consider $\mathcal{K}$ as a multiset in $\mathbb{Z}_{N_k}$ with constant multiplicity p_k. Recall that our proof of (10.4) works regardless of the parity of M. Hence, if $\mathcal{K}$ is not fibered on a $D(N_k)$-grid Λ, it must satisfy one of the conclusions of Lemma 8.6 with $N=N_k$ on that grid. Since a corner structure is prohibited by Lemma 10.6, it remains to verify that $\mathcal{K}\cap\Lambda$ mod $\mathbb{Z}_{N_k}$ cannot be a set of diagonal boxes.

Given $a\in\mathcal{K}$, we identify the grid $\Lambda_0=\Lambda(a,D(N_k))$ with $\mathbb{Z}_{p_i}\times\mathbb{Z}_{p_j}\times\mathbb{Z}_{p_k}$, where for every $x\in\Lambda_0$ we have $\mathbb{A}^{N_k}_{N_k}[x]\in\{0,p_k\}$. Note that under this representation, elements that differ only in their p_k coordinate are at distance N_k/p_k from one another. Assume, by contradiction, that $\mathcal{K}\cap\Lambda_0$ is a set of diagonal boxes $(I\times J\times K)\cup(I^c\times J^c\times K^c)$ with multipliicty p_k, and let

$$a\in I\times J\times K. \tag{10.6}$$

Since $p_i=2$, we may assume without loss of generality that $I=\{0\},I^c=\{1\}$.

Claim 1. Suppose that $\mathcal{K}\cap\Lambda_0$ is a set of diagonal boxes as above. Then

(i) $\max\{|J||K|,|J^c||K^c|\}\le p_j$,
(ii) $\max\{|K|,|K^c|\}\le 2$.

Proof. For (i), assuming the contrary, we get

$$|A\cap\Pi(a,p_i^2)|\ge p_k|I\times J\times K|=p_k|J||K|>p_jp_k,$$

which contradicts Lemma 7.3.

For (ii), if $|K| > 2$, then

$$|A \cap \Pi(a, p_j^2)| \geq \mathbb{K}_{N_k}^{N_k}[a] + \mathbb{K}_{N_k/p_k}^{N_k}[a] > p_k|K| > p_i p_k,$$

again contradicting Lemma 7.3.

Applying the same argument with a replaced by an element of $I^c \times J^c \times K^c$, we get $|J^c||K^c| \leq p_j$ and $|K^c| \leq 2$. □

Claim 1(i) implies that $p_k = |K| + |K^c| \leq 4$, hence $p_k = 3$ and $p_j \geq 5$. We assume this from here on, with $|K| = 2, |K^c| = 1$. Then

$$|A \cap \ell_k(a)| = p_k|K| = p_i p_k, \tag{10.7}$$

with $A \cap \ell_k(a) = \mathcal{K} \cap \ell_k(a)$. By Lemma 7.3, $|A \cap \Pi(a, p_j^2)| = p_i p_k$ and

$$(A \cap \Pi(a, p_j^2)) \subset \ell_k(a). \tag{10.8}$$

By Claim 1 (i), we have $|J| \leq p_j/2$ and therefore

$$|J^c| \geq p_j/2 > 2. \tag{10.9}$$

Claim 2. Suppose that $\mathcal{K} \cap \Lambda_0$ is a set of diagonal boxes as above. Then $\Phi_{p_i^2}|A$.

Proof. We have

$$\begin{aligned} |A \cap \Pi(a, p_i)| &\geq p_k(|K||J| + |K^c||J^c|) \\ &= p_k(2|J| + |J^c|) > p_j p_k. \end{aligned}$$

By Corollary 7.4, this implies the claim. □

The next claim contradicts Claim 2, proving that $\mathcal{K} \cap \Lambda_0$ cannot be a set of diagonal boxes.

Claim 3. Suppose that $\mathcal{K} \cap \Lambda_0$ is a set of diagonal boxes as above. Then $\Phi_{p_i^2}|B$.

Proof. Let $x \in I^c \times J \times K^c$ with $(x - a, M) = M/p_i p_k^2$, where a is as in (10.6). By (10.8), we have $x \in \mathbb{Z}_M \setminus A$. Consider the saturating set A_x. By (10.9), we have $A_x \subset \Pi(x, p_j^2) = \Pi(a, p_j^2)$. By (10.8), we further have $A_x \subset \ell_k(a)$. As in (10.7), we see that $A \cap \ell_k(a) = \mathcal{K} \cap \ell_k(a)$, so that $\mathbb{A}_{M/p_i p_k}[x] = 0$ and

$$\mathbb{A}_{M/p_i p_k^2}[x|\ell_k(a)]\mathbb{B}_{M/p_i p_k^2}[b] = \phi(p_i p_k^2) \text{ for any } b \in B.$$

With $p_k = 3$ and $p_i = 2$, we have $\phi(p_i p_k^2) = 6 = p_i p_k = \mathbb{A}_{M/p_i p_k^2}[x|\ell_k(a)]$, so that

$$\mathbb{B}_{M/p_i p_k^2}[b] = 1.$$

Since $|K| = 2$, we also have $M/p_k, M/p_k^2 \in \text{Div}(A)$, so that $\mathbb{B}_{M_k}^{M_k}[b] = 1$ for any $b \in B$. These, in turn, imply that the set B is organized in pairs so that for every $b \in B$ we have

$$B \cap \ell_k(b) = \{b\}$$

and there exists a unique $b' \in B$ with $(b-b', M) = M/p_i p_k^2$, so that $B \cap \Lambda(b, M/p_i p_k^2) = \{b, b'\}$. Since any plane $\Pi(y, p_i)$ with $y \in \mathbb{Z}_M$ may be written as a disjoint union of nonintersecting $M/p_i p_k^2$-grids, and the intersection of B with each such grid is either empty or a two-point set as above, we see that

$$|B \cap \Pi(y, p_i^2)| = \frac{1}{p_i}|B \cap \Pi(y, p_i)| \quad \forall y \in \mathbb{Z}_M,$$

so that $\Phi_{p_i^2}|B$. □

It follows that $\mathcal{K}\cap\Lambda_0$ must be N_k-fibered in some direction. This together with part (i) of the corollary implies (ii). □

The next lemma complements Lemma 10.7 in the case when M is even.

Lemma 10.9. *Assume (F) with $p_i=2$. Suppose that $\Phi_{N_i}|A$ and that there exists a $D(N_i)$-grid Λ on which $\mathcal{I}$ is not fibered. Then $\mathcal{I}\cap\Lambda$ mod N_i contains diagonal boxes $(I\times J\times K)\cup(I^c\times J^c\times K^c)$, $\Phi_{p_j^2}\Phi_{p_k^2}|A$, and for any $x\in(I\times J^c\times K^c)\cup(I^c\times J\times K)$ we have*

$$\mathbb{I}^{N_i}_{N_i/p_i}[x]=\phi(p_i^2)=p_i. \tag{10.10}$$

Moreover:

(i) *If $\min(|J|,|K|)\geq 2$ or $\min(|J^c|,|K^c|)\geq 2$, then*

$$\{D(M)|m|M\}\cup\{M/p_i^2p_jp_k\}\subset \mathrm{Div}(A\cap\Lambda). \tag{10.11}$$

(ii) *If $|J|=|K^c|=1$, then*

$$\{M/p_i,M/p_j,M/p_k,M/p_ip_j,M/p_ip_k,M/p_i^2p_jp_k\}\subset \mathrm{Div}(A\cap\Lambda), \tag{10.12}$$

and for any $x\in\mathbb{Z}_{p_i}\times J\times K^c$,

$$\begin{aligned}|A\cap\Pi(x,p_j^2)|&=p_i(p_k-1),\\ |A\cap\Pi(x,p_k^2)|&=p_i(p_j-1).\end{aligned} \tag{10.13}$$

If $|J^c|=|K|=1$, the same conclusions hold with the j and k indices interchanged.

Proof. By Lemma 10.6, $\mathcal{I}\cap\Lambda$ mod N_i cannot have an extended corner structure. Therefore, by Proposition 8.5, it must contain a pair of diagonal boxes $(I\times J\times K)\cup(I^c\times J^c\times K^c)$, with each point of multiplicity p_i. We note that

$$\begin{aligned}|A\cap\Pi(x,p_j)|&\geq p_i(|I||J||K|+|I^c||J^c||K^c|)\\ &=p_i(|J||K|+|J^c||K^c|)\\ &>p_ip_k\end{aligned}$$

It follows from Corollary 7.4 that $\Phi_{p_j^2}|A$. The same argument, with j and k interchanged, proves that $\Phi_{p_k^2}|A$. In addition, any $x\in(I\times J^c\times K^c)\cup(I^c\times J\times K)$ satisfies (10.10).

For the moreover part, assume first that $\min(|J|,|K|)\geq 2$. Then clearly

$$\{D(M)|m|M\}\subset \mathrm{Div}(I\times J\times K)\subset \mathrm{Div}(A),$$

and

$$M/p_i^2p_jp_k\in \mathrm{Div}(I\times J\times K, I^c\times J^c\times K^c)\subset \mathrm{Div}(A). \tag{10.14}$$

Thus (10.11) is proved.

Next, assume $|J|=|K^c|=1$. Then

$$\{M/p_i,M/p_k,M/p_ip_k\}\subset \mathrm{Div}(I\times J\times K)\subset \mathrm{Div}(A),$$

$$\{M/p_i,M/p_j,M/p_ip_j\}\subset \mathrm{Div}(I^c\times J^c\times K^c)\subset \mathrm{Div}(A),$$

and (10.14) holds here as well. Finally, (10.13) follows from the diagonal boxes structure. □

Recall that $M_\nu = M/p_\nu^2$ has only two distinct prime factors for each $\nu \in \{i,j,k\}$. In particular, all M_ν-cuboids are 2-dimensional, so that Lemma 7.6 applies on that scale. Thus, if $\Phi_{M_i}|A$, then A mod M_i is a linear combination of M_i-fibers in the p_j and p_k directions, with non-negative integer coefficients. In particular, if $\Phi_{M_i}|A$ and

$$\mathbb{A}^{M_i}_{M_i}[x] \in \{0, c_0\} \quad \forall x \in \mathbb{Z}_M,$$

then A is M_i-fibered in one of the p_j and p_k directions on every $D(M_i)$-grid. Similar statements hold with A replaced by B, as well as for other permutations of the indices i,j,k. In particular, we have the following fibering result.

Lemma 10.10. *Assume that (F) holds. Suppose $\Phi_{M_k}|A$, then for every $a_k \in \mathcal{K}$ we have $\mathbb{K}^{M_k}_{M_k}[a_k] = p_k$. Moreover, we must have one of the following:*

$$a_k \text{ belongs to an } M_k\text{-fiber in the } p_i \text{ direction, i.e. } \mathbb{K}^{M_k}_{M_k/p_i}[a_k] = p_k\phi(p_i), \tag{10.15}$$

or

$$a_k \text{ belongs to an } M_k\text{-fiber in the } p_j \text{ direction, i.e. } \mathbb{K}^{M_k}_{M_k/p_j}[a_k] = p_k\phi(p_j). \tag{10.16}$$

In addition, if (10.15) holds then

$$|A \cap \Pi(a_k, p_j^2)| = p_ip_k \text{ and } (A \cap \Pi(a_k, p_j^2)) \subset \mathcal{K}, \tag{10.17}$$

and if (10.16) holds then $|A \cap \Pi(a_k, p_i^2)| = p_jp_k$ and $(A \cap \Pi(a_k, p_i^2)) \subset \mathcal{K}$.

Proof. We write $\mathcal{K}$ mod M_k as a linear combination of M_k-fibers in the p_i and p_j directions with non-negative integer coefficients, as permitted by Lemma 7.6 with $N = M_k$. Then for every $a_k \in \mathcal{K}$ we have $\mathbb{K}^{M_k}_{M_k}[a_k] = (c_i + c_j)p_k$, where c_ν is the number of M_k-fibers in the p_ν direction passing through a_k in that representation. Without loss of generality assume that $c_i > 0$. It follows that

$$\begin{aligned} |A \cap \Pi(a_k, p_j^2)| &\geq \mathbb{K}^{M_k}_{M_k}[a_k] + \mathbb{K}^{M_k}_{M_k/p_i}[a_k] \\ &\geq (c_i + c_j)p_k + c_ip_k\phi(p_i) \\ &\geq p_ip_k \end{aligned}$$

Recall from Lemma 7.3 that $p_ip_k \geq |A \cap \Pi(a_k, p_j^2)|$, so that the chain of inequalities above must holds with equalities. In particular, $\mathbb{K}^{M_k}_{M_k}[a_k] = c_i = 1$. □

11. Splitting for fibered grids

In this section, we assume the following.

Assumption (F'): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i}p_j^{n_j}p_k^{n_k}$ has three distinct prime factors. (We do not need to assume that $n_i = n_j = n_k = 2$.) Furthermore, $\Phi_M|A$, and A is fibered on $D(M)$-grids.

Our goal is to establish preliminary structure of $A \oplus B$ on $D(M)$-grids under the assumption (F'). We are interested in how the fibers of $\mathcal{I}, \mathcal{J}, \mathcal{K}$ (defined as in Theorem 10.1) can be combined in a tiling. More specifically, let Λ be an N-grid for some $N|M$ such that $D(M)|N$. We will say that:

- Λ is *tiled by M-fibers in the p_i direction* if $\Sigma_A(\Lambda) \subset \mathcal{I}$,
- Λ is *tiled by M-fibers in the p_i and p_j directions* if $\Sigma_A(\Lambda) \subset \mathcal{I} \cup \mathcal{J}$,

and similarly for other permutations of the indices i, j, k. Note that if Λ is tiled by M-fibers in the p_i direction, this does not exclude the possibility that $\Sigma_A(\Lambda)$ may include fibers in other directions: for example, some element $a \in \Sigma_A(\Lambda)$ could belong to both $\mathcal{I}$ and $\mathcal{K}$. We will prove that every $D(M)$-grid is tiled by M-fibers in at most 2 directions, with the "stratified" structure described in Lemma 11.2. We then prove a few results concerning the localization of $\Sigma_A(\Lambda)$, in the same spirit as Lemma 4.7, but with stronger conclusions since we now have the fibering assumption at our disposal.

The results here, with the exception of Lemma 11.2, will not be used in our proof of Theorem 10.1. However, they provide additional structural information that does not follow from either (T2) or Theorem 10.1, and they apply in the more general case when n_i, n_j, n_k need not be equal to 2.

First, we note the following simple argument for future reference.

Lemma 11.1. *Assume (F'), and let $z, z' \in \mathbb{Z}_M$ with $z = a + b$, $z' = z' + b'$, $a, a' \in A$, $b, b' \in B$. If $(z - z', M) = M/p_i$ and $a \in \mathcal{I}$, then $a' \in a * F_i$ and $b = b'$. In particular, $a' \in \mathcal{I}$.*

Proof. We have $z' - z = \nu M/p_i$ for some $\nu \in \{0, 1, \ldots, p_i - 1\}$. Hence $z' = z + \nu M/p_i = (a + \nu M/p_i) + b$, with $a + \nu M/p_i \in a * F_j \subset A$. By uniqueness, $a' = a + \nu M/p_i \in \mathcal{I}$. $\square$

We now prove our structure result for $D(M)$-grids. The geometric intuition here is provided by tilings of a 3-dimensional space by rectangular columns. Define $R_i, R_j, R_k \subset \mathbb{R}^3$ by

$$R_i = \mathbb{R} \times [0,1] \times [0,1], \quad R_j = [0,1] \times \mathbb{R} \times [0,1], \quad R_k = [0,1] \times [0,1] \times \mathbb{R}.$$

Then any tiling of $\mathbb{R}^3$ by translated, pairwise disjoint (up to sets of measure zero) copies of R_i, R_j, R_k must use translated copies of at most two of these sets. Moreover, the tiling can be stratified into "slabs" of thickness 1, with each slab tiled by translates of just one of R_i, R_j, R_k. We note, however, that we must also consider cases when M-fibers in two or three directions may intersect.

Lemma 11.2. (Two directions in a grid) *Assume (F'), and let $\Lambda = \Lambda(z_0, D(M))$ for some $z_0 \in \mathbb{Z}_M$. Then, possibly after a permutation of the indices i, j, k, the following holds.*

(i) **(p_k-stratified structure)** *For $\nu = 0, 1, \ldots, p_k - 1$, let $z_\nu = z_0 + \nu M/p_k$ and $\Lambda_\nu = z_\nu * F_i * F_j$. Then $\{0, 1, \ldots, p_k - 1\}$ can be partitioned into disjoint sets $\mathcal{D}_i$ and $\mathcal{D}_j$ so that*

$$\Sigma_A(\Lambda_\nu) \subset \mathcal{I} \text{ for all } \nu \in \mathcal{D}_i, \tag{11.1}$$

$$\Sigma_A(\Lambda_\nu) \subset \mathcal{J} \text{ for all } \nu \in \mathcal{D}_j. \tag{11.2}$$

Note that one of $\mathcal{D}_i$ and $\mathcal{D}_j$ may be empty. Moreover, $\mathcal{D}_i$ and $\mathcal{D}_j$ are not always uniquely determined: for example, if $\Sigma_A(\Lambda_\nu) \subset \mathcal{I} \cap \mathcal{J}$ for some ν, then we may assign that ν to either $\mathcal{D}_i$ or $\mathcal{D}_j$.

(ii) *Assume, in addition, that*

$$\Sigma_A(\Lambda) \cap \mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset, \tag{11.3}$$

and that we may choose $\mathcal{D}_i \neq \emptyset$ and $\mathcal{D}_j \neq \emptyset$. Then $\Sigma_A(\Lambda) \cap \mathcal{K} = \emptyset$.

(iii) *Suppose that*

$$\Sigma_A(\Lambda) \cap \mathcal{I} \cap \mathcal{J} \cap \mathcal{K} \neq \emptyset, \tag{11.4}$$

and that there exists $a \in \Sigma_A(\Lambda) \cap \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$ such that $A \cap \Lambda(a, D(M))$ is M-fibered in the p_i direction. Then $\Sigma_A(\Lambda) \subset \mathcal{I}$.

We remark that parts (i) and (iii) of the lemma include the case considered in [25, Lemma 9.5 and Corollary 9.6]. Since the proof is short, we include it here for completeness.

Proof. We first note that if the conclusion (i) of the lemma holds for some z_0, then it also holds with z_0 replaced by any other $z \in \Lambda$, with the same relabelling of i, j, k, and with the $\mathcal{D}_i$ and $\mathcal{D}_j$ sets adjusted appropriately. This will allow us to move z_0 to any other point of the same grid.

We first prove that (i) holds for some permutation of the indices i, j, k. This will also include the proof of (iii).

Case 1. Suppose that (11.4) holds. Choose $z \in \Lambda$ such that $z = a + b$, where $a \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, $b \in B$, and $A \cap \Lambda(a, D(M))$ is M-fibered in the p_i direction. Then

$$A_0 := (a * F_i * F_j) \cup (a * F_i * F_k) \subset A.$$

We will prove that $\Sigma_A(\Lambda) \subset \mathcal{I}$. To this end, we claim the following: for $z' \in \Lambda$, let $a_{z'} \in A, b_{z'} \in B$ satisfy $a_{z'} + b_{z'} = z'$. Define

$$Z_1 := \{z' \in \Lambda : \ b_{z'} = b\}, \ Z_2 := \Lambda \setminus Z_1.$$

Then

$$\text{if } z' \in Z_2, \text{ then } a_{z'} \in \mathcal{I} \text{ and } z' * F_i \subset Z_2. \tag{11.5}$$

Assuming (11.5), we complete the proof as follows. By (11.5), we have $\Sigma_A(Z_2) \subset \mathcal{I}$. Moreover, $Z_2 = \bigcup_{z' \in Z_2} z' * F_i$ is a union of M-fibers in the p_i direction. Therefore, so is Z_1, as its complement in Λ. But $\Sigma_A(Z_1) = (-b) * Z_1$, so that $\Sigma_A(Z_1) \subset \mathcal{I}$ as well. Hence $\Sigma_A(\Lambda) \subset \mathcal{I}$ as claimed. This clearly implies (i) and (iii).

We now prove (11.5). It suffices to prove that for any $z' \in F_2$, and for any $z_i \in z' * F_i$ (with $z_i = z'$ permitted), we cannot have $a_{z_i} \in \mathcal{J} \cup \mathcal{K}$. Indeed, suppose that $a_{z_i} \in \mathcal{J}$, and let $(z * F_i * F_k) \cap (z_i * F_j) = \{z_j\}$. Then $z_j = a_{z_j} + b$ for some $a_{z_j} \in a * F_i * F_k$. By Lemma 11.1, $a_{z_j} \in a_{z_i} * F_j$ and $b_{z_i} = b$. Applying Lemma 11.1 again, this time to z_i and z', we get $b_{z'} = b_{z_i} = b$, contradicting the assumption that $z' \in F_2$. The case when $a' \in \mathcal{K}$ is similar.

Case 2. Assume now that (11.3) holds, and that there exists $z \in \Lambda$ such that $z = a + b$, where $b \in B$ and a belongs to two of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$. Without loss of generality, we may assume that $a \in \mathcal{I} \cap \mathcal{J}$, so that $a * F_i * F_j \subset A$. Then the set $\Lambda_0 := z * F_i * F_j$ satisfies both (11.1) and (11.2).

Define z_ν and Λ_ν as in the statement of the lemma, with $z_0 = z$. Let $\nu \neq 0$. Assume that $z_\nu = a_\nu + b_\nu$, where $a_\nu \in A$ and $b_\nu \in B$. If we had $a_\nu \in \mathcal{K}$, Lemma 11.1 would imply $a \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, contradicting (10.1). Hence a_ν belongs to one or both of $\mathcal{I}$ and $\mathcal{J}$.

If $a_\nu \in \mathcal{I} \cap \mathcal{J}$, then $\Lambda_\nu = (a_\nu + b_\nu) * F_i * F_j$, and both (11.1) and (11.2) hold. Suppose now that $a_\nu \in \mathcal{I} \setminus \mathcal{J}$, and let $z_{\nu,\mu} = z_\nu + \mu M/p_j$ for $\mu = 0, 1, \ldots, p_j - 1$. Let $z_{\nu,\mu} = a_{\nu,\mu} + b_{\nu,\mu}$ with $a_{\nu,\mu} \in A$, $b_{\nu,\mu} \in B$.

- If we had $a_{\nu,\mu} \in \mathcal{K}$ for any μ, an application of Lemma 11.1 to $z_{\nu,\mu}$ and the point at the intersection of $\ell_k(z_{\nu,\mu})$ and Λ_0 would contradict (10.1).
- If we had $a_{\nu,\mu} \in \mathcal{J}$, we could apply Lemma 11.1 to $z_{\nu,\mu}$ and z_ν to prove that $a_\nu \in \mathcal{J}$, contradicting the assumption that $a_\nu \in \mathcal{I} \setminus \mathcal{J}$.

It follows that $a_{\nu,\mu} \in \mathcal{I}$ for all μ, and (11.1) holds for this ν. If on the other hand $a_\nu \in \mathcal{J} \setminus \mathcal{I}$, then (11.2) holds, with the same proof except that the i, j indices are interchanged.

Case 3. Assume now that for each $z \in \Lambda$, we have $z = a + b$, where $b \in B$ and a belongs to exactly one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$. Without loss of generality, assume that $z_0 = a_0 + b_0$, where $b_0 \in B$ and $a_0 \in \mathcal{I}$. Define z_ν and Λ_ν as in the lemma, and assume that $z_\nu = a_\nu + b_\nu$, where $a_\nu \in A$ and $b_\nu \in B$. By Lemma 11.1 we cannot have $a_\nu \in \mathcal{K}$ for any ν, since that would imply $a_0 \in \mathcal{I} \cap \mathcal{K}$, contradicting the assumption of Case 3.

- Suppose that there exists a ν such that $a_\nu \in \mathcal{J}$. Let $z'_{\nu,\mu} = z_\nu + \mu M/p_i = (z_0 + \mu M/p_i) + \nu M/p_j$ for $\mu = 0, 1, \dots, p_i - 1$. Suppose that $z'_{\nu,\mu} = a'_{\nu,\mu} + b'_{\nu,\mu}$ with $a'_{\nu,\mu} \in A$, $b'_{\nu,\mu} \in B$. Applying Lemma 11.1 twice (once in the p_i direction to $z'_{\nu,\mu}$ and z_ν, and once in the p_k direction to $z'_{\nu,\mu}$ and $z_0 + \mu M/p_i$), we see that $a'_{\nu,\mu} \notin \mathcal{I} \cup \mathcal{K}$. Therefore $a'_{\nu,\mu} \in \mathcal{J}$ for all μ, and (11.2) holds for that ν. Continuing the proof as in Case 2, we see that the conclusion of the lemma holds.
- Suppose now that $a_\nu \in \mathcal{I}$ for all ν. Then $\Sigma_A(z_0 * F_i * F_k) \subset \mathcal{I}$, and by the same argument as above, $\Sigma_A(\Lambda)$ has a p_j-stratified structure, as in (i) but with the j and k indices interchanged.

It remains to prove (ii). Suppose that $\Sigma_A(\Lambda)$ has a p_k-stratified structure, with $\mathcal{D}_i$ and $\mathcal{D}_j$ both nonempty, and assume by contradiction that there exists an element $a \in \Sigma_A(\Lambda) \cap \mathcal{K}$. Let $z \in \Lambda$ be a point such that $a + b = z$ for some $b \in B$. Without loss of generality, we may assume that $z \in \Lambda_\nu$ for some $\nu \in \mathcal{D}_i$. Then $a \in \mathcal{I} \cap \mathcal{K}$, so that $a * F_i * F_k \subset A$. Since $\mathcal{D}_j \neq \emptyset$, there exists a point $z' \in z * F_k$ such that $z' \in \Lambda_{\nu'}$ for some $\nu' \in \mathcal{D}_j$. Then $z' = a' + b$, where $a' = a + (z' - z) \in a * F_k$. By the choice of ν', we have $a' \in \mathcal{J}$. However, we also have $a' \in a * F_i * F_k \subset A$, so that $a' \in \mathcal{I} \cap \mathcal{K}$. This $a' \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, contradicting the assumptions of the lemma. □

Lemma 11.3. (No cross-direction fibers) *Let $M = p_1^{n_1} \dots p_K^{n_K}$. Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling. Let $\Lambda_{ij} := \Lambda(z, M/p_ip_j)$ for some $z \in \mathbb{Z}_M$ and $i, j \in \{1, \dots, K\}$. Suppose that $a_0 * F_i \subset A$ for some $a_0 \in \Sigma_A(\Lambda_{ij})$. Then*

$$\Sigma_A(\Lambda_{ij}) \subset \Pi(a_0, p_j^{n_j-1}). \tag{11.6}$$

Proof. Let $b_0 \in B$ satisfy $z_0 := a_0 + b_0 \in \Lambda_{ij}$. By translational invariance, we may assume that $z_0 = a_0 = b_0 = 0$, so that $F_i \subset A$. Then $\Lambda_{ij} = \bigcup_{a \in F_i} a * F_j$. For each $a \in F_i$, we have $a = a + 0$, with $a \in A$ and $0 \in B$. Applying Lemma 4.5 to $a * F_j$, we see that $\Sigma_A(a * F_j) \subset \Pi(a_0, p_j^{n_j-1})$. But $a \in F_i$ was arbitrary, and (11.6) follows. □

Proposition 11.4. (Consistency in fibered grids) *Assume that (F') holds, and that $0 \in A \cap B$. Suppose that $\Lambda := \Lambda(0, D(M))$ is M-fibered in the p_i direction. Then:*

(i) *There exists $l \in \{j, k\}$ such that*

$$\Sigma_A(\Lambda) \subset \Pi(0, p_l^{n_l-1}). \tag{11.7}$$

(ii) *Assume further that* $\Sigma_A(\Lambda) \subset \mathcal{I} \cup \mathcal{J}$, *with both* $\Sigma_A(\Lambda) \cap \mathcal{I} \neq \emptyset$ *and* $\Sigma_A(\Lambda) \cap \mathcal{J} \neq \emptyset$. *Then (11.7) holds with* $l = k$.

Proof. We first claim that $\Sigma_A(\Lambda)$ has either a p_j or a p_k stratified structure as in Lemma 11.2 (i). Indeed, suppose that we are in the complementary case when $\Sigma_A(\Lambda)$ has a p_i-stratified structure. We prove that it also has a stratified structure in one of the other directions.

- Assume that (11.3) holds. Since $\Sigma_A(\Lambda) \cap \mathcal{I} \neq \emptyset$, Lemma 11.2 (ii) implies that one of $\mathcal{D}_j$ and $\mathcal{D}_k$ is empty. Without loss of generality, assume that $\mathcal{D}_k = \emptyset$. But then $\Sigma_A(\Lambda) \subset \mathcal{J}$, so that $\Sigma_A(\Lambda)$ has the p_k-stratified structure with $\mathcal{D}_i = \emptyset$.
- Assume now that (11.4) holds. Then $\Sigma_A(\Lambda)$ is contained in just one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$. If $\Sigma_A(\Lambda)$ is contained in $\mathcal{I}$ or $\mathcal{J}$, it has a p_k-stratified structure; otherwise, it has a p_j-stratified structure.

Assume, without loss of generality, that $\Sigma_A(\Lambda)$ has a p_k-stratified structure, with $z_0 = 0$ and $\Sigma_A(\Lambda) \subset \mathcal{I} \cup \mathcal{J}$. Let $\Lambda_{ij} = F_i * F_j$ and $\Lambda_{ik} = F_i * F_k$. We first claim that

$$\Sigma_A(\Lambda_{ij}) \subset \mathcal{I}. \tag{11.8}$$

Indeed, if $0 \in \mathcal{D}_i$, we are done. If on the other hand $0 \in \mathcal{D}_j$, then $\Sigma_A(\Lambda_{ij}) \subset \mathcal{J}$. In particular, $a \in \mathcal{J}$ for all $a \in F_i$, so that $F_i * F_j \subset A$ and $\Sigma_A(\Lambda_{ij}) = F_i * F_j \subset \mathcal{I}$, as claimed.

Proof of (i). By Lemma 11.3, we have

$$\Sigma_A(\Lambda_{ij}) \subset \Pi(0, p_j^{n_j-1}), \quad \Sigma_A(\Lambda_{ik}) \subset \Pi(0, p_k^{n_k-1}).$$

It suffices to prove that, additionally, we have

$$\text{either } \Sigma_A(\Lambda_{ij}) \subset \Pi(0, p_k^{n_k-1}) \text{ or } \Sigma_A(\Lambda_{ik}) \subset \Pi(0, p_j^{n_j-1}). \tag{11.9}$$

Indeed, suppose that the first inclusion in (11.9) holds. Writing $\Lambda = \bigcup_{z\in\Lambda_{ij}} z * F_k$ and applying Lemma 4.5 to each $z * F_k$, we get that (11.7) holds with $l = k$. Assuming the second inclusion in (11.9), we get the same with $l = j$.

It remains to prove (11.9) Assume, by contradiction, that

$$\Sigma_A(\Lambda_{ij}) \not\subset \Pi(0, p_k^{n_k-1}) \text{ and } \Sigma_A(\Lambda_{ik}) \not\subset \Pi(0, p_j^{n_j-1}).$$

Then there must exist $z_j \in \Lambda_{ij}, z_k \in \Lambda_{ik}$, and $a_j, a_k \in A, b_j, b_k \in B$ satisfying

$$a_\nu + b_\nu = z_\nu,\ \nu \in \{j, k\}.$$

such that

$$p_j^{n_j-1} \nmid a_k, b_k, \quad p_k^{n_k-1} \nmid a_j, b_j. \tag{11.10}$$

By (11.8), we have $a_j \in \mathcal{I}$. Replacing a_j by a different point of $a_j * F_i$ if necessary, we may assume that $z_j \in \Pi(z_k, p_i^{n_i})$. Let $z_0 \in \Lambda$ and $a_i \in F_i$ satisfy

$$(z_0 - z_j, M) = (a_i - z_k, M) = M/p_k, \quad (z_0 - z_k, M) = (a_i - z_j, M) = M/p_j.$$

Let $a_0 \in A, b_0 \in B$ satisfy $z_0 = a_0 + b_0$. By Lemma 4.5 and (11.10),

$$p_j^{n_j-1} \nmid a_k - a_0,\ p_k^{n_k-1} \nmid a_j - a_0.$$

On the other hand, since $a_i = a_i + 0 \in \Sigma_A(F_i)$, it follows from Lemma 4.7 that $p_\nu^{n_\nu-1}$ must divide a_0 for some $\nu \in \{j, k\}$. This contradiction proves (i).

Proof of (ii). Let $z_i, z_j \in \Lambda$ satisfy $z_i = a_i + b_i$, $z_j = a_j + b_j$ with $a_i \in \mathcal{I}, a_j \in \mathcal{J}, b_1, b_2 \in B$. We claim that

$$p_k^{n_k-1} | a_i - a_j. \tag{11.11}$$

To prove this, let $a_i' \in a_i * F_i$ and $a_j' \in a_j * F_j$, to be fixed later. Let

$$(a_i' - a_j', M) = p_i^{\alpha_i} p_j^{\alpha_j} p_k^{\alpha_k}, \quad (b_j' - b_i', M) = p_i^{\beta_i} p_j^{\beta_j} p_k^{\beta_k}.$$

Suppose that $a_i' - a_j'$ and $b_j' - b_j$ do not match in the p_i direction, so that $\alpha_i \neq \beta_i$. By Lemma 4.3 (iii), we must have $\{\alpha_i, \beta_i\} = \{n_i - 1, n_i\}$. But then we can move a_i' to a different point of F_i to get $\alpha_i = \beta_i$. Similarly, replacing a_j' by a different point of $a_j * F_j$ if needed, we can ensure that $\alpha_j = \beta_j$, getting a match in the p_j direction. Hence $a_i' - a_j'$ and $b_j' - b_j$ cannot match in the p_k direction. By Lemma 4.3 (iii) again, this implies (11.11).

With (11.11) in place, the proof of (ii) is completed as follows. Suppose that A satisfies the assumptions of (ii). Then (11.11) implies the following: given $a_i \in \Sigma_A(\Lambda) \cap \mathcal{I}$, any $a_j \in \Sigma_A(\Lambda) \cap \mathcal{J}$ must satisfy $a_j \in \Pi(a_i, p_k^{n_k-1})$, and the same holds with the i and j indices interchanged. Since $\Sigma_A(\Lambda) \cap \mathcal{I}$ and $\Sigma_A(\Lambda) \cap \mathcal{J}$ are both nonempty, this clearly implies (11.7) with $l = k$. □

We briefly discuss splitting parity and the slab reduction for fibered grids. If $A \oplus B = \mathbb{Z}_M$, where M has at most 3 prime factors, our expectation is that the condition of the slab reductions are satisfied for at least one of A and B, in some direction [24, Conjecture 9.1]. By Lemma 5.6, this means uniform splitting parity in that direction for all tilings $A \oplus rB = \mathbb{Z}_M$ with $r \in R$. An intermediate result in this direction might establish uniform splitting parity on certain subgrids of $\mathbb{Z}_M$. For example, the following is easy to prove.

Lemma 11.5. *Assume that (F') holds. Let $\Lambda = \Lambda(z_0, D(M))$ for some $z_0 \in \mathbb{Z}_M$. Assume that $\Sigma_A(\Lambda)$ has a p_i-stratified structure as in Lemma 11.2 (i), and that*

$$|\mathcal{D}_i| \geq 2 \text{ and } |\mathcal{D}_j| \geq 2. \tag{11.12}$$

Then the tiling has uniform splitting parity in the p_k direction on Λ. (That is, all fibers $z * F_k$ with $z \in \Lambda$ split with the same parity, either (A, B) for all z or (B, A) for all z).

Proof. We first prove the following.

$$\text{for all } z \in \Lambda,\ z * F_i * F_k \text{ has uniform splitting parity in the } p_k \text{ direction.}$$

We may assume that $z = z_0$ is the point specified in Lemma 11.2 (see the remark at the beginning of the proof of the lemma). Suppose that $\nu, \mu \in \mathcal{D}_i$ with $\nu \neq \mu$, and let $z_\nu = z + \nu M/p_k, z_\mu = z + \mu M/p_k$. Then $z_\nu = a_\nu + b_\nu$ and $z_\mu = a_\mu + b_\mu$, with $a_\nu, a_\mu \in \mathcal{I}$.

Let $x \in z * F_i$, so that $x = z + \lambda M/p_i$ for some $\lambda \in \{0, 1, \dots, p_i - 1\}$. Consider the points $x_\nu, x_\mu \in x * F_k$ given by $x_\nu = z_\nu + \lambda M/p_i$ and $x_\mu = z_\mu + \lambda M/p_i$. We have

$$x_\nu = (a_\nu + \lambda M/p_i) + b_\nu, \quad x_\mu = (a_\mu + \lambda M/p_i) + b_\mu.$$

Thus $z * F_k$ splits with parity (B, A) if and only if $p_k^{n_k} | b_\nu - b_\mu$, if and only if $x * F_k$ splits with parity (B, A). The case of parity (A, B) is similar.

We now complete the proof of the lemma. Let $z_0 \in \Lambda$. By (11.12), $z * F_k$ splits with the same parity as $z_0 * F_k$ for all $z \in z_0 * F_i$. By (11.12) again, applied to $z \in z_0 * F_i$ with i and j interchanged, $z' * F_k$ splits with the same parity as $z * F_k$ for all $z' \in z * F_j$. But this means that all fibers $z' * F_k$ with $z' \in z_0 * F_i * F_j$ split with the same parity, as claimed. □

We believe that, at least for $n_i = n_j = 2$, Lemma 11.5 can be proved using our current methods without the assumption that (11.12) holds. However, our tentative argument to that end is much longer and more involved. Since the result is not needed in our proof of (T2), we do not include it here.

12. Proof of Theorem 10.1, part (II a)

12.1. **Preliminary results.** We first point out a special case when Theorem 10.1 (II a) has a short proof based on Lemma 11.2. Both Lemma 12.1 and Corollary 12.2 are valid under more general assumptions on the cardinalities of A and B than (F). In particular, Corollary 12.2 assumes that $|A| = p_i p_j p_k$, but does not require the same of B.

Lemma 12.1. *Assume that $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, and that A is fibered on $D(M)$-grids. Assume further that*

$$\{D(M)|m|M\} \cap \mathrm{Div}(B) = \{M\}. \tag{12.1}$$

Then:

(i) *For any $D(M)$-grid Λ, we have $|\Sigma_A(\Lambda)| = |\Lambda|$.*
(ii) *If $|A| = |\Lambda| = p_i p_j p_k$, then A is contained in the union of just two of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$.*

Proof. We first prove (i). For each $z \in \Lambda$, let $a_z \in A$ and $b_z \in B$ satisfy $z = a_z + b_z$. Suppose that $a_z = a_{z'}$ for some $z, z' \in \Lambda$ with $z \neq z'$. Then $b_z - b_{z'} = (z - a_z) - (z' - a_{z'}) = z - z'$, so that $b_z \neq b_{z'}$ and $D(M)|(b_z - b_{z'})$. But that contradicts (12.1).

If $|A| = p_i p_j p_k$, then (i) implies that $\Sigma_A(\Lambda) = A$. By Lemma 11.2, $\Sigma_A(\Lambda)$ may be written as a union of fibers in just two directions. This implies (ii). □

Corollary 12.2. *Assume that $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, and that $|A| = p_i p_j p_k$, $\Phi_M|A$, and A is fibered on $D(M)$-grids. Assume further that (10.1) and (12.1) hold. Then at least one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty.*

Proof. Let $\Lambda := \Lambda(0, D(M))$. By Lemma 12.1, we have $\Sigma_A(\Lambda) = A$. After a permutation of the i, j, k indices if necessary, we may assume that $\Sigma_A(\Lambda)$ has the structure given in Lemma 11.2, so that $A \subset \mathcal{I} \cup \mathcal{J}$.

If $\mathcal{I} = \emptyset$ or $\mathcal{J} = \emptyset$, we are done. Assume therefore that there exist $z_i, z_j \in \Lambda$ such that $z_i = a_i + b_i, z_j = a_j + b_j$ with $a_i \in \mathcal{I}, a_j \in \mathcal{J}$, and $b_i, b_j \in B$. Assume also, for contradiction, that $\mathcal{K} \neq \emptyset$. Since $A \subset \mathcal{I} \cup \mathcal{J}$, we may assume without loss of generality that there exists an element $a_k \in \mathcal{I} \cap \mathcal{K}$. By Lemma 10.2, we have $a_k * F_i * F_k \subset A$. In particular, there exists $z \in \Lambda(z_j, M/p_i p_j)$ such that $z = a + b$ for some $a \in a_k * F_i * F_k$. Then $a \in \mathcal{I} \cap \mathcal{K}$; but also, by the choice of z_j, we have $a \in \mathcal{J}$. This contradicts (10.1). □

Next, suppose that (F) holds, and that $\Phi_{N_\nu}|A$ and A mod N_ν has an unfibered $D(N_\nu)$-grid. By Corollary 10.8, this cannot happen when $p_\nu > \min(p_i, p_j, p_k)$. When $p_\nu = \min(p_i, p_j, p_k)$, such unfibered grids are permitted and must have one of the structures described in Lemmas 10.7 and 10.9. In this case, however, Theorem 10.1 (II a) also has a short proof, based on Corollary 12.2.

Proposition 12.3. *Assume that (F) and (10.1) hold. Assume further that there exists $\nu \in \{i, j, k\}$ such that $\Phi_{N_\nu}|A$ and A mod N_ν has an unfibered $D(N_\nu)$-grid Λ. Then one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty.*

Proof. Suppose first that $A \cap \Lambda$ has either one of the structures in Lemma 10.7 with $p_k = \min p_\nu$, or the structure in Lemma 10.9 (i). In both of these cases, we have $\{D(M)|m|M\} \subset \mathrm{Div}(A)$, so that in particular (12.1) holds. The conclusion, in those cases, follows now from Corollary 12.2. Lemma 12.4 takes care of the last remaining case. □

Lemma 12.4. *Assume (F) with $p_i = 2$. Suppose that $\Phi_{N_i}|A$, and that there exists a $D(N_i)$-grid Λ on which $\mathcal{I}$ mod N_i has the structure given in Lemma 10.9 (ii). Assume further that (10.1) holds. Then at least one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty.*

Proof. We will use a splitting argument to prove that (12.1) must also hold in this case. With that established, the lemma follows from Corollary 12.2.

We first recall the structure given in Lemma 10.9 (ii). Specifically, $\mathcal{I} \cap \Lambda$ mod N_i consists of diagonal boxes $(I \times J \times K) \cup (I^c \times J^c \times K^c)$ in $\mathbb{Z}_{N_i}$, where each point has multiplicity 2 in $\mathbb{Z}_{N_i}$ (coming from an M-fiber in the p_i direction in $\mathbb{Z}_M$). Without loss of generality, we may assume that $2 = p_i < p_k < p_j$. Relabelling I, J, K as I^c, J^c, K^c and vice versa if necessary, we may also assume that

$$|J| = |K^c| = 1.$$

We recall that, by (10.12),

$$\{M/p_i, M/p_j, M/p_k, M/p_ip_j, M/p_ip_k, M/p_i^2p_jp_k\} \subset \mathrm{Div}(A \cap \Lambda).$$

Assume, by contradiction, that $m \in \{D(M)|m|M\} \cap \mathrm{Div}(B)$ and $m \neq M$. By (10.12), we have $m \in \{M/p_jp_k, M/p_ip_jp_k\}$. Suppose that $b, b' \in B$ satisfy $(b - b', M) = m$. By translational invariance, we may assume that $b' = 0$ and that $0 \in I \times J \times K$.

Let $\Lambda = \Lambda(0, D(M))$. Let $G \subset \Lambda$ be the set that projects to $I \times J \times K$ modulo N_i, so that G is the union of $p_k - 1$ disjoint M-fibers in the p_i direction. Consider the sets G and $b * G$, with b as above. Since $p_k \geq 3$, we must have $M/p_j \in \mathrm{Div}(G, b * G)$. In other words, there exist $z, z' \in \Lambda$ such that $(z - z', M) = M/p_j$ and $z = a + b$, $z' = a' + b' = a' + 0$, with $a, a' \in G$. It follows that $z * F_j$ splits with parity (A, B), so that for every $z'' \in z * F_j$ we have $z'' = a'' + b''$ for some $a'' \in A \cap \Pi(0, p_j^2)$ and $b'' \in B$.

On the other hand, if $z_1, z_2 \in z * F_j$ satisfy $z_1 \neq z_2$ and $z_\nu = a_\nu + b_\nu$ for some $a_\nu \in A \cap \Pi(0, p_j^2)$ and $b_\nu \in B$, we cannot have $M/p_i|(a_1 - a_2)$. Indeed, otherwise $b_1 - b_2 = (z_1 - z_2) - (a_1 - a_2)$ would be divisible by M/p_i, contradicting (10.12). Therefore at most $p_k - 1$ points of $z * F_j$ can be tiled by points of G. Since $p_j - (p_k - 1) \geq 3$, there are at least 3 more points of $z * F_j$ that need to be tiled some other way, each one requiring a different point of $A \cap \Pi(0, p_j^2)$. However, $|G| = p_i(p_k - 1)$, so that by Lemma 7.3, $(A \cap \Pi(0, p_j^2)) \setminus G$ may contain at most 2 distinct points. This contradiction proves the lemma. □

12.2. **Case (F1).** We recall the assumption (F1) for the reader's convenience.

Assumption (F1): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2p_j^2p_k^2$. Furthermore, $|A| = |B| = p_ip_jp_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, (10.1) holds, and $\Phi_{M_\nu}|A$ for some $\nu \in \{i, j, k\}$.

Our goal is to prove the following.

Proposition 12.5. *Assume that (F1) holds. Then one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty.*

Without loss of generality we shall assume that

$$\Phi_{M_k}|A. \tag{12.2}$$

Assume for contradiction that the proposition fails, so that

$$\mathcal{I}, \mathcal{J}, \mathcal{K} \neq \emptyset. \tag{12.3}$$

We will prove that this is impossible.

Lemma 12.6. *Assume that (F1), (12.2), and (12.3) hold. If there exists $a_k \in \mathcal{K}$ satisfying (10.15), then $\Phi_{p_j^2} \nmid A$. The same holds with i and j interchanged.*

Proof. Assume for contradiction that (10.15) holds for some $a_k \in \mathcal{K}$, but $\Phi_{p_j^2}|A$. The second assumption, together with (10.17), implies that $A \subset \Pi(a_k, p_j)$.

Suppose that $a_j \in \mathcal{J}$. Then $a_j * F_j \subset \Pi(a_k, p_j)$, and there exists an element $a_j' \in (a_j * F_j) \cap \Pi(a_k, p_j^2)$. By (10.17), we have $a_j' \in \mathcal{J} \cap \mathcal{K}$, so that

$$a_j * F_j * F_k = a_j' * F_j * F_k = A \cap \Pi(a_j, p_i^2) \tag{12.4}$$

by Lemma 10.2. In particular, $a_j \in \mathcal{K}$. Since $a_j \in \mathcal{J}$ was arbitrary, we have $\mathcal{J} \subset \mathcal{K}$.

Additionally, (10.15) and (12.4) imply that $|A \cap \Pi(a_j, p_i^2)| = p_j p_k$ and $|A \cap \Pi(a_j, p_i)| > p_j p_k$. By Corollary 7.4, we have $\Phi_{p_i^2}|A$ and $A \subset \Pi(a_j, p_i)$.

Now let $a_i \in \mathcal{I}$. By (12.4) and Lemma 7.3, there must be an element $a_i' \in a_i * F_i$ such that $a_i' \in a_j * F_j * F_k$. But then $a_i' \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, contradicting (10.1). □

Lemma 12.7. *Assume that (F1), (12.2) and (12.3) hold. If there exists $a_k \in \mathcal{K}$ satisfying (10.15), then the same holds for all $a \in \mathcal{K}$. Moreover, $\Phi_{p_j}|A$. The same is true with the i and j indices interchanged.*

Proof. Assume for contradiction that there exist $a, a' \in \mathcal{K}$ such that a satisfies (10.15) and a' satisfies (10.16). We also assume, without loss of generality, that $p_i > p_j$.

From (10.15) and (10.17) we see that

$$|A \cap \Pi(a, p_i)| \geq p_i p_k > p_j p_k.$$

By Corollary 7.4, we get $\Phi_{p_i^2}|A$. This, however, implies by Lemma 12.6 that (10.16) cannot hold for a', a contradiction. The second conclusion follows from Lemma 12.6. □

In light of Lemmas 12.6 and 12.7, we may assume from now on that

$$\text{(10.15) holds for all } a \in \mathcal{K}, \text{ and } \Phi_{p_j}|A. \tag{12.5}$$

We record the following simple consequence.

Corollary 12.8. *Assume that (F1), (12.2), (12.3) and (12.5) hold. Then:*

(i) $|A \cap \Pi(a_k, p_i)| \geq p_i p_k$ *for all* $a_k \in \mathcal{K}$,
(ii) $|A \cap \Pi(x, p_j)| = p_i p_k$ *for all* $x \in \mathbb{Z}_M$,
(iii) $\mathcal{K} \cap \Pi(a_j, p_j) = \emptyset$ *for all* $a_j \in \mathcal{J}$ *(in particular,* $\mathcal{J} \cap \mathcal{K} = \emptyset$*)*,
(iv) $(\mathcal{I} \setminus \mathcal{J}) \cap \Pi(a_j, p_j) \neq \emptyset$ *for all* $a_j \in \mathcal{J}$,
(v) $p_j < p_k$.

Proof. Parts (i) and (ii) follow directly from, respectively, the first and second part of (12.5).

We now prove (iii). Let $a_j \in \mathcal{J}$, and assume for contradiction that there exists an element $a_k \in \mathcal{K} \cap \Pi(a_j, p_j)$. By (10.17), we have $|A \cap \Pi(a_k, p_j^2)| = p_i p_k$. Since $a_j * F_j$ cannot all be contained in $\Pi(a_k, p_j^2)$, we get $|A \cap \Pi(a_j, p_j)| > p_i p_k$, contradicting (ii).

By (iii), we have $A\cap\Pi(a_j,p_j)\subset\mathcal{I}\cup\mathcal{J}$. It follows that there exist integers $c_j>0, c_i\geq 0$ such that

$$p_ip_k=|A\cap\Pi(a_j,p_j)|=c_ip_i+c_jp_j,$$

with c_ip_i accounting for the elements of $(\mathcal{I}\setminus\mathcal{J})\cap\Pi(a_j,p_j)$. Hence $p_k=c_i+c_j'p_j$ for some $c_j'>0$, and $p_j<p_k$ as claimed in (v). Furthermore, we must have $c_i>0$, since p_j does not divide p_k. This proves (iv). □

Lemma 12.9. *Assume that (F1), (12.2), (12.3), and (12.5) hold. Then $\Phi_{M_i}\nmid A$.*

Proof. Assume, by contradiction, that $\Phi_{M_i}|A$. By Lemma 10.3, we have $\Phi_{M_i}|\mathcal{I}$.

Let $a_j\in\mathcal{J}$. By Corollary 12.8 (iii), $A\cap\Pi(a_j,p_j)\subset\mathcal{I}\cup\mathcal{J}$. By Lemma 10.10 with M_k replaced by M_i, each $a_i\in\mathcal{I}\cap\Pi(a_j,p_j)$ must satisfy either

$$\mathbb{I}^{M_i}_{M_i/p_j}[a_i]=p_i\phi(p_j). \tag{12.6}$$

or

$$\mathbb{I}^{M_i}_{M_i/p_k}[a_i]=p_i\phi(p_k) \tag{12.7}$$

Suppose that (12.7) holds for some $a_i\in\mathcal{I}\cap\Pi(a_j,p_j)$. Then $|\mathcal{I}\cap\Pi(a_i,p_j^2)|=p_ip_k$, so that

$$|A\cap\Pi(a_j,p_j)|\geq|\mathcal{I}\cap\Pi(a_i,p_j^2)|+|(a_j*F_j)\setminus\{a_j\}|>p_ip_k,$$

contradicting Corollary 12.8 (ii).

Hence (12.6) holds for all $a_i\in\mathcal{I}\cap\Pi(a_j,p_j)$. We conclude that there must be two nonnegative integers c_i,c_j such that

$$p_ip_k=|A\cap\Pi(a_j,p_j)|=c_ip_ip_j+c_jp_j,$$

with the first term accounting for all a_i as above and the second term accounting for all $a_j\in(\mathcal{J}\setminus\mathcal{I})\cap\Pi(a_j,p_j)$. This, however, is not allowed since p_j does not divide p_ip_k. □

The main step in the proof of Proposition 12.5 is the following proposition.

Proposition 12.10. *Assume that (F1), (12.2), (12.3) and (12.5) hold. Then*

$$\Phi_{N_i}\nmid A. \tag{12.8}$$

We first finish the proof of Proposition 12.5, assuming Proposition 12.10.

Lemma 12.11. *Assume that (F1), (12.2), (12.3), and (12.5) hold. Then B is N_i-fibered in the p_j direction, with $\mathbb{B}_{M/p_ip_j}[b]=\phi(p_j)$ for all $b\in B$. Consequently, $p_j=\min_\nu p_\nu$, $M/p_ip_j\in\mathrm{Div}(B)$, and $\mathcal{I}\cap\mathcal{J}=\emptyset$.*

Proof. Consider B modulo N_i. Since $M/p_i\notin\mathrm{Div}(B)$, we have

$$\mathbb{B}^{N_i}_{N_i}[y]=\mathbb{B}_M[y]+\mathbb{B}_{M/p_i}[y]\in\{0,1\}\text{ for all }y\in\mathbb{Z}_M.$$

By Lemma 12.9 and (12.8), we have $\Phi_{N_i}\Phi_{M_i}|B$. Furthermore, since $M/p_i\in\mathrm{Div}(A)$, (7.5) holds with $c_0=1$. It follows from Corollary 7.7 that B is a union of pairwise disjoint N_i-fibers in the p_j and p_k directions.

It remains to prove that B mod N_i cannot contain an N_i-fiber in the p_k direction. Indeed, assume for contradiction that $\{b_1,\dots,b_{p_k}\}$ is such a fiber. Since $M/p_k\in\mathrm{Div}(A)$, we must have $(b_\mu-b_\nu,M)=M/p_ip_k$ for $\mu\neq\nu$. This is only possible if $p_k<p_i$. However, consider

any M_k-fiber in the p_i direction in $\mathcal{K}$, as provided by (12.5). If $p_k < p_i$, then any such fiber must include M/p_ip_k as a difference. Thus $M/p_ip_k \in \mathrm{Div}(A) \cap \mathrm{Div}(B)$, a contradiction. □

By Corollary 12.8 and Lemma 12.11, we have $|A \cap (a_k, p_i)| \geq p_ip_k > p_jp_k$. Corollary 7.4 implies that

$$\Phi_{p_i^2}|A. \tag{12.9}$$

Lemma 12.12. *Assume (F1), (12.2), (12.3), and (12.5). Then* $\Phi_{M_j} \nmid A$.

Proof. Assume, by contradiction, that $\Phi_{M_j}|A$. By Lemma 10.3, this implies that $\Phi_{M_j}|\mathcal{J}$. By Lemma 10.10 with k and j interchanged, $\mathcal{J}$ mod M_j is a union of disjoint M_j-fibers in the p_i and p_k directions. Suppose first that there exists an M_j-fiber in the p_i direction in $\mathcal{J}$, with $\mathbb{J}^{M_j}_{M_j/p_i}[a_j] = p_j\phi(p_i)$ for some $a_j \in \mathcal{J}$. Since $p_j < p_i$ by Lemma 12.11, it follows that $M/p_ip_j \in \mathrm{Div}(A)$; but this also contradicts Lemma 12.11.

Hence $\mathbb{J}^{M_j}_{M_j/p_k}[a_j] = p_j\phi(p_k)$ for all $a_j \in \mathcal{J}$. In particular, $|\mathcal{J} \cap \Pi(a_j, p_i^2)| \geq p_jp_k$, and by Lemma 7.3, the latter holds with equality. This together with (12.9) implies that $A \subset \Pi(a_j, p_i)$.

Now, let $a_k \in \mathcal{K}$. Since $a_k \in \Pi(a_j, p_i)$ and satisfies (10.15), there must be an element $a'_k \in \mathcal{K} \cap \Pi(a_j, p_i^2)$. Since $a'_k \notin \mathcal{J}$ by Corollary 12.8 (iii), we get $|A \cap \Pi(a_j, p_i^2)| > p_jp_k$, contradicting Lemma 7.3. □

Lemma 12.13. *Assume (F1), (12.2), (12.3), and (12.5). Then* $\Phi_{N_j} \nmid A$.

Proof. Let $a_j \in \mathcal{J}$. By Corollary 12.8 (iii) and Lemma 12.11, we have $a_j \notin \mathcal{I} \cup \mathcal{K}$. Hence $A \cap \Lambda(a_j, D(M)) \subset \mathcal{J}$, and there exist $x_i, x_k \in \mathbb{Z}_M \setminus A$ such that $(a_j - x_\nu, M) = M/p_\nu$ and $\mathbb{J}^{N_j}_{N_j}[a_\nu] = 0$ for $\nu \in \{i, k\}$.

We have $\Phi_{N_j}|A$ if and only if $\Phi_{N_j}|\mathcal{J}$. Consider the evaluation of $\mathcal{J}$ on an N_j-cuboid with one face containing vertices at a_j, x_i, and x_k, and the other face in $\Pi(a_k, p_j)$ for some $a_k \in \mathcal{K}$. In order to balance that cuboid, $\mathcal{J} \cap \Pi(a_k, p_j)$ must be nonempty. But then $\mathcal{J} \cap \Pi(a_k, p_j^2)$ is nonempty, contradicting (10.17). □

Proof of Proposition 12.5. By Lemmas 12.12 and 12.13, we have $\Phi_{N_j}\Phi_{M_j}|B$. It follows from Corollary 7.7, with $c_0 = 1$ since $M/p_j \in \mathrm{Div}(A)$, that B is a union of pairwise disjoint N_j-fibers in the p_i and p_k directions.

Let $\nu \in \{i, k\}$, and suppose that $\{b_1, \ldots, b_{p_\nu}\}$ is a N_j-fiber in the p_ν direction. Since $M/p_\nu \in \mathrm{Div}(A)$, we must have $(b_\mu - b_{\mu'}, M) = M/p_jp_\nu$ for all $\mu \neq \mu'$. However, this is not possible, since $p_j = \min_\nu p_\nu$ by Lemma 12.11. This contradiction concludes the proof of the proposition. □

Proof of Proposition 12.10. The proof is divided into several steps. In each of the following claims, the assumptions of the proposition are assumed to hold. We will also assume, by contradiction, that (12.8) does not hold, so that $\Phi_{N_i}|A$. By Lemma 10.3, this implies that

$$\Phi_{N_i}|\mathcal{I}.$$

Claim 1. *Let* $a_j \in \mathcal{J}$. *Then:*

(i) *There must exist* $a_i \in \mathcal{I} \cap \Pi(a_j, p_j)$ *such that*

$$\ell_i(a_i) \subset A, \text{ and } a_i \notin \mathcal{J}. \tag{12.10}$$

(ii) *Consequently,* $p_i = \min_\nu p_\nu$.

Proof. Let $a_i \in (\mathcal{I} \setminus \mathcal{J}) \cap \Pi(a_j, p_j)$, as provided by Corollary 12.8 (iv). By Proposition 12.3, the grid $\Lambda(a_i, D(N_i))$ must be N_i-fibered in some direction. However, fibering in the p_ν direction for some $\nu \in \{j, k\}$ would imply that $a_i * F_i * F_\nu \subset A$; for $\nu = j$, this is impossible by the assumption that $a_i \in \mathcal{I} \setminus \mathcal{J}$, and for $\nu = k$, this is prohibited by Corollary 12.8 (iii).

It follows that any $a_i \in (\mathcal{I} \setminus \mathcal{J}) \cap \Pi(a_j, p_j)$ must belong to an N_i-fiber in the p_i direction, so that (12.10) holds. This, moreover, implies that $|A \cap \ell_i(a_i)| = p_i^2$. If $p_i > p_j$ or $p_i > p_k$, this contradicts Lemma 7.3. □

Let $a_j \in \mathcal{J}$, and let $a_i \in \mathcal{I} \cap \Pi(a_j, p_j)$ satisfy (12.10). Replacing a_j by another element of $a_j * F_j$, and a_i by another element of $\ell_i(a_i)$, if necessary, we may assume without loss of generality that $M_k | a_i - a_j$.

Claim 2. *With* a_i *and* a_k *as above, we have*

$$(a_i - a_j, M) = M/p_k^2. \tag{12.11}$$

Proof. Assume for contradiction that (12.11) fails, so that $(a_i - a_j, M) = M/p_k$. Then $a_i \in \Lambda := \Lambda(a_j, D(M))$. The grid Λ contains no elements of $\mathcal{K}$ by Corollary 12.8 (iii), hence it cannot be M-fibered in the p_k direction. It cannot be fibered in the p_j direction, either, since $a_i \notin \mathcal{J}$. It must be, therefore, M-fibered in the p_i direction, so that

$$a_j * F_i * F_j \subset A, \tag{12.12}$$

and $|A \cap \Pi(a_j, p_k)| > p_i p_j$ since $a_i \notin a_j * F_i * F_j$. It follows by Corollary 7.4 that $\Phi_{p_k^2} | A$ and $A \subset \Pi(a_j, p_k)$. However, this implies that $\mathcal{K} \subset \Pi(a_j, p_k)$, hence $\mathcal{K} \cap \Pi(a_j, p_k^2) \neq \emptyset$. Thus $\Pi(a_j, p_k^2)$ must contain the $p_i p_j$ points in (12.12), and at least one additional point of $\mathcal{K}$ which, by Corollary 12.8 (iii), does not belong to $a_j * F_i * F_j$. This violates Lemma 7.3. □

Claim 2, together with (12.3), yields the following (partial) list of divisors in A:

$$M/p_i, M/p_j, M/p_k, M/p_i^2, M/p_k^2, M/p_i p_k^2, M/p_i^2 p_k^2 \in \mathrm{Div}(A) \tag{12.13}$$

Claim 3. *B is N_k-fibered in the p_j direction, with* $\mathbb{B}_{M/p_j p_k}[b] = \phi(p_j)$ *for all* $b \in B$.

Proof. We claim that

$$M/p_i p_k \notin \mathrm{Div}(A) \tag{12.14}$$

Indeed, if (12.14) were not true, then this together with (12.13) would imply

$$|B \cap \Pi(b, p_j^2)| \leq p_i \text{ for all } b \in B.$$

By Corollary 12.8 (v),

$$|B| \leq p_i p_j^2 < p_i p_j p_k,$$

a contradiction.

Next, let $a_k \in \mathcal{K}$. By (12.5), a_k satisfies (10.15), and due to (12.14) we must have

$$\mathbb{K}_{M/p_ip_k^2}[a_k] = p_k\phi(p_i). \tag{12.15}$$

Fix $b \in B$ and $x \in \mathbb{Z}_M$ with $(x - a_k, M) = M/p_j$, and consider the saturating set $A_{x,b}$. Recall from Corollary 12.8 (ii) that $A \cap \Pi(a_k, p_j) \subset \Pi(a_k, p_j^2)$, hence $A_{x,b} \subset \Pi(a_k, p_j^2)$. Together with (12.14) and (12.15), the latter implies

$$\begin{aligned}
1 &= \langle \mathbb{A}[x], \mathbb{B}[b] \rangle \\
&= \frac{1}{\phi(p_jp_k)} \mathbb{A}_{M/p_jp_k}[x|\Pi(a_k, p_j^2)] \mathbb{B}_{M/p_jp_k}[b] + \frac{1}{\phi(p_ip_jp_k^2)} \mathbb{A}_{M/p_ip_jp_k^2}[x|\Pi(a_k, p_j^2)] \mathbb{B}_{M/p_ip_jp_k^2}[b] \\
&= \frac{1}{\phi(p_j)} \mathbb{B}_{M/p_jp_k}[b] + \frac{1}{\phi(p_jp_k)} \mathbb{B}_{M/p_ip_jp_k^2}[b] \\
&= \frac{1}{\phi(p_j)} \sum_{(y-b,M)=M/p_j} \left(\mathbb{B}_{M/p_k}[y] + \frac{1}{\phi(p_k)} \mathbb{B}_{M/p_ip_k^2}[y] \right)
\end{aligned}$$

Since $M/p_ip_k^2 \in \mathrm{Div}(A)$ by (12.13), Lemma 7.1 implies that

$$\mathbb{B}_{M/p_k}[y] \cdot \mathbb{B}_{M/p_ip_k^2}[y] = 0, \text{ for all } y \in \mathbb{Z}_M. \tag{12.16}$$

On the other hand, again by (12.13), we have

$$\mathbb{B}_{M/p_ip_k^2}[y] \leq \phi(p_i) < \phi(p_k), \tag{12.17}$$

where at the last step we used Claim 1 (ii). Given (12.16) and (12.17), the only way to saturate $\langle \mathbb{A}[x], \mathbb{B}[b] \rangle$ is to have $\mathbb{B}_{M/p_jp_k}[b] = \phi(p_j)$ as claimed. □

At this point, it may be useful to pause and consider the geometric meaning of what we have proved so far, in the framework of Lemma 11.2. Suppose that $0 \in A \cap B$, with $0 \in \mathcal{J}$. By Lemma 11.2, the grid $\Lambda := \Lambda(0, D(M))$ must be tiled by fibers in at most 2 directions, with the structure indicated by the lemma. However, the subgrid $F_j * F_k$ cannot be tiled solely by fibers of $\mathcal{J}$, since by Claim 3 each such fiber would tile p_j fibers of $F_j * F_k$ and p_k is not divisible by p_j. Therefore $\Sigma_A(\Lambda) \subset \mathcal{I} \cup \mathcal{J}$, with $(\mathcal{I} \setminus \mathcal{J}) \cap \Sigma_A(\Lambda) \neq \emptyset$. We will see that this forces B to have very strong fibering properties, which eventually become incompatible with each other.

We now return to the proof of the proposition.

Claim 4. *We have*

$$\Phi_{N_j} \nmid A.$$

Proof. Assume by contradiction that $\Phi_{N_j}|A$. By Lemma 10.3, $\Phi_{N_j}|\mathcal{J}$. By Claim 3, Lemma 7.8, and Corollary 10.8 (i), $\mathcal{J}$ is N_j-fibered in the p_i direction, so that $\mathcal{J} \subset \mathcal{I}$. We prove that this is not allowed.

Let $a_j \in \mathcal{J}$, so that $a_j * F_i * F_j \subset A$. By Lemma 7.3, $A \cap \Pi(a_j, p_k^2) = a_j * F_i * F_j$. If we had $\Phi_{p_k^2}|A$, then A would be contained in $\Pi(a_j, p_k)$; this, however, contradicts the conclusion of Claim 2. It follows that $\Phi_{p_k}|A$, $\Phi_{p_k^2}|B$, and

$$A \cap \Pi(a_j, p_k) = a_j * F_i * F_j. \tag{12.18}$$

Let $x \in \mathbb{Z}_M$ with $(x - a_j, M) = M/p_k$ and consider the saturating set A_{x,b_0}, where $b_0 \in B$ is arbitrary. By (12.18), for every $y \in \mathbb{Z}_M$ with $(y - b_0, M) = M/p_k$ we have

$$1 = \mathbb{B}_M[y] + \mathbb{B}_{M/p_i}[y] + \mathbb{B}_{M/p_j}[y] + \mathbb{B}_{M/p_ip_j}[y].$$

This implies that $B \cap \Lambda(b_0, D(M)) = \{b_0, b_1, b_2, \ldots, b_{p_k-1}\}$, where $p_k \parallel b_\mu - b_{\mu'}$ for all $\mu, \mu' \in \{0, 1, \ldots, p_k - 1\}, \mu \neq \mu'$. On the other hand, taking Claim 3 into account, this p_k-tuple of elements $b_0, \ldots, b_{p_k-1}$ can be grouped into pairwise disjoint N_k-fibers in the p_j direction, each of cardinality p_j. This implies p_k is divisible by p_j, which is not allowed. □

Claim 5. *Given $a_i \in \mathcal{I}$ satisfying (12.10), we have $\mathbb{I}^{M_i}_{M_i/p_\mu}[a_i] = 0$ for $\mu \in \{j, k\}$.*

Proof. Let a_i satisfy (12.10). Let $b, b' \in B$ with $(b - b', M) = M/p_jp_k$, as follows from Claim 3. Let $y, y' \in \mathbb{Z}_M \setminus B$ with $(b-y, M) = (b'-y', M) = M/p_k$, $(b-y', M) = (b'-y, M) = M/p_j$, and consider the saturating set B_{y,a_i}. Then

$$B_{y,a_i} \subset \ell_i(b) \cup \ell_i(y) \cup \ell_i(b') \cup \ell_i(y').$$

If $B_{y,a_i} \cap (\ell_i(b) \cup \ell_i(b'))$ were nonempty, then $\{M/p_i, M/p_i^2\} \cap \mathrm{Div}(B)$ would be nonempty, contradicting (12.13). Thus $B_{y,a_i} \subset (\ell_i(y) \cup \ell_i(y'))$. It follows that $\{M/p_i^\delta p_j, M/p_i^\delta p_k\} \subset \mathrm{Div}(B)$ for some $\delta \in \{1, 2\}$. By (12.10), in both cases we get $\mathbb{I}^{M_i}_{M_i/p_\mu}[a_i] = 0$ for $\mu \in \{j, k\}$. □

The next three claims are identical to Claims 7, 8, and 9 in the proof of Proposition 9.14 of [25]. The proofs are exactly the same as in [25], and are therefore omitted. The only difference is that, in the proof of Claim 6, we have to start by choosing a_i satisfying (12.10).

Claim 6. *We have $\Phi_{M_i/p_j}\Phi_{M_i/p_k}|B$.*

Claim 7. *For each $\mu \in \{j, k\}$, B is M_i-fibered in the p_μ direction, with $\mathbb{B}^{M_i}_{M_i/p_\mu}[b] = \phi(p_\mu)$ for each $b \in B$.*

Claim 8. *B is N_j-fibered in the p_i direction, with $\mathbb{B}^M_{M/p_ip_j}[b] = \phi(p_i)$ for each $b \in B$.*

We are now in a position to finish the proof of Proposition 12.10. Fix $b \in B$. By Claims 1 and 7, $\mathbb{B}^{M_i}_{M_i}[b] = 1$ and $\mathbb{B}^{M_i}_{M_i/p_j}[b] = \phi(p_j)$. Thus

$$|B \cap \Lambda(b, M_i/p_j)| = p_j \text{ for all } b \in B.$$

On the other hand, we can write $\Lambda(b, M_i/p_j)$ as a union of pairwise disjoint grids $\Lambda(y_\nu, M/p_ip_j)$ for an appropriate choice of y_ν. By Claim 8, each such grid contains either 0 or p_i elements of B. But this implies that p_i divides p_j, a contradiction. □

12.3. **Case (F2).** In this section, we consider the following case.

Assumption (F2): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2p_j^2p_k^2$. Furthermore, $|A| = |B| = p_ip_jp_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, (10.1) holds, and $\Phi_{M_\nu} \nmid A$ for all $\nu \in \{i, j, k\}$.

Proposition 12.14. *Assume that (F2) holds. Then one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty.*

Assume, for contradiction, that (12.3) holds. Without loss of generality, we may also assume that

$$p_i < p_j < p_k. \tag{12.19}$$

Lemma 12.15. *Assume that (F2), (12.3), and (12.19) hold. Then neither Φ_{N_j} nor Φ_{N_k} divides A.*

Proof. By Corollary 10.8 and (12.19), if $\Phi_{N_j}|A$, then $\mathcal{J} \subset \mathcal{I} \cup \mathcal{K}$. However, that would imply $\Phi_{M_j}|A$, contradicting (F2). The same argument holds with j and k interchanged. $\square$

Corollary 12.16. *Assume that (F2), (12.3), and (12.19) hold. Then $\Phi_{N_j}\Phi_{M_j}|B$. Consequently, B is N_j-fibered in the p_i direction, with*

$$\mathbb{B}_{M/p_ip_j}[b] = \phi(p_i) \text{ for all } b \in B. \tag{12.20}$$

Moreover, we have $\Phi_{p_i}|A$ and $\Phi_{p_i^2}|B$.

Proof. The first part follows from (F2) and Lemma 12.15. This, in turn, implies by Corollary 7.7 (with $c_0 = 1$, since $N_j \in \text{Div}(A)$) that B is a union of pairwise disjoint N_j-fibers in the p_i and p_k directions. We also have $p_j < p_k$ by (12.19), so that having an N_j-fiber in the p_k direction in B would imply that $N_j \in \text{Div}(B)$, a contradiction. This proves the fibering claim. Finally, (12.20) and Lemma 7.2 imply that $\Phi_{p_i^2}|B$ as claimed. $\square$

Proposition 12.17. *Assume that (F2), (12.3), and (12.19) hold. Then $\Phi_{N_i} \nmid A$.*

As in Section 12.3, we first finish the proof of Proposition 12.14, assuming Proposition 12.17.

Proof of Proposition 12.14. By (F2) and Proposition 12.17, we have $\Phi_{N_i}\Phi_{M_i}|B$. Applying Corollary 7.7 with $c_0 = 1$ once more, we see that B is a union of pairwise disjoint N_j-fibers in the p_i and p_k directions, so that for every $b \in B$ we must have

$$\mathbb{B}_{M/p_ip_\nu}[b] = \phi(p_\nu) \text{ for some } \nu = \nu(b) \in \{j, k\}.$$

By (12.19), this implies $M/p_\nu \in \text{Div}(A) \cap \text{Div}(B)$ for at least one $\nu \in \{j, k\}$, which is a contradiction. $\square$

Proof of Proposition 12.17. Again, we split the proof into several steps. In each of the following claims, the assumptions of the proposition are assumed to hold. We will also assume, by contradiction, that $\Phi_{N_i}|A$. By Lemma 10.3, this implies that $\Phi_{N_i}|\mathcal{I}$.

Claim 1. *$\mathcal{I}$ is a union of pairwise disjoint N_i-fibers in the p_i and p_k directions. Moreover, $\mathbb{I}^{N_i}_{N_i/p_i}[a_i] = \phi(p_i^2)$ must hold for at least one element $a_i \in \mathcal{I}$, and $M/p_i^2 \in \text{Div}(A)$.*

Proof. By (12.20), we have $M/p_ip_j \notin \text{Div}(A)$, therefore $N_i/p_j \notin \text{Div}_{N_i}(\mathcal{I})$. Moreover, $\mathbb{I}^{N_i}_{N_i}[x] \in \{0, p_i\}$ for all $x \in \mathbb{Z}_M$. It follows from Lemma 7.8 that $\mathcal{I}$ mod N_i is a union of pairwise disjoint N_i-fibers, with multiplicity 2, in the p_i and p_k directions. This implies the first part of the lemma.

If $\mathcal{I}$ were N_i-fibered in the p_k direction, this would imply $\mathcal{I} \subset \mathcal{K}$. Then, however, we would have $\Phi_{M_i}|A$, contradicting (F2). Hence at least one $a_i \in \mathcal{I}$ must belong to an N_i-fiber in the p_i direction, as claimed. $\square$

Claim 2. *B is M_i-fibered in the p_k direction, with $\mathbb{B}^{M_i}_{M_i/p_k}[b] = \phi(p_k)$ for each $b \in B$. Consequently,*

$$M/p_ip_k \in \text{Div}(B). \tag{12.21}$$

Moreover:

(i) *There is no element $b \in B$ satisfying $\mathbb{B}^{M_i}_{M_i/p_j}[b] = \phi(p_j)$.*
(ii) *$\mathcal{I}$ is N_i-fibered in the p_i direction.*

Proof. We have $\Phi_{M_i}|B$ by (F2). This means that B satisfies the assumptions of Lemma 10.10, with $\mathcal{K}$ replaced by B, the p_k and p_i directions interchanged, and with multiplicity 1 (instead of p_k) by Claim 1. It follows that B is a union of pairwise disjoint M_i-fibers in the p_j and p_k directions.

We now argue as in the proof of Proposition 12.10. Assume for contradiction that (i) fails, so that some $b \in B$ belongs to an M_i-fiber in the p_j direction, with $\mathbb{B}^{M_i}_{M_i/p_j}[b] = \phi(p_j)$. This means that

$$|B \cap \Lambda(b, M_i/p_j)| = p_j \text{ for all } b \in B.$$

On the other hand, we can write $\Lambda(b, M_i/p_j)$ as a union of pairwise disjoint M/p_ip_j-grids. By (12.20), each such grid contains either 0 or p_i elements of B. But then $p_i|p_j$, which is obviously false. This proves (i), and the fibering claim for B follows. Part (ii) now follows from (12.21) and Claim 1. □

Claim 3. *There exists $b_0 \in B$ such that $\mathbb{B}_{M/p_jp_k}[b_0] = \phi(p_j)$.*

Proof. By (F2) and Lemma 12.15, we have $\Phi_{N_k}\Phi_{M_k}|B$. It follows from Corollary 7.7 with $c_0 = 1$ that B is a union of pairwise disjoint N_k-fibers in the p_i and p_j directions, so that for every $b \in B$ we have $\mathbb{B}_{M/p_\mu p_k}[b] = \phi(p_\mu)$ for some $\mu \in \{i, j\}$. However, if the latter was true with $\mu = i$ for all $b \in B$, then Claim 2 and the divisibility argument in its proof would show that $p_i|p_k$. This contradiction proves the claim. □

We note that (12.20), (12.21), and Claim 3 show that $\{M/p_ip_j, M/p_ip_k, M/p_jp_k\} \subset \mathrm{Div}(B)$. Hence $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ are pairwise disjoint.

The next two claims are proved in [25], Claim 7 and 8 of Proposition 9.14. The proof is identical.

Claim 4. $\Phi_{M_i/p_j}\Phi_{M_i/p_k}|B$.

Claim 5. *B is M_i-fibered in both of the p_j and p_k directions, so that for all $b \in B$ we have*

$$\frac{1}{\phi(p_j)}\mathbb{B}^{M_i}_{M_i/p_j}[b] = \frac{1}{\phi(p_k)}\mathbb{B}^{M_i}_{M_i/p_k}[b] = 1.$$

Since Claim 5 is in direct contradiction with Claim 2 (i), the proposition follows. □

13. Proof of Theorem 10.1, part (II c)

In this section, we will work under the following assumption.

Assumption (F3): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2p_j^2p_k^2$. Furthermore, $|A| = |B| = p_ip_jp_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, $\mathcal{I} = \emptyset$, and

$$\text{the sets } \mathcal{J} \setminus \mathcal{K} \text{ and } \mathcal{K} \setminus \mathcal{J} \text{ are nonempty.} \tag{13.1}$$

Proposition 13.1. *Assume (F3). Then the conclusion (II c) of Theorem 10.1 holds.*

The proof below works regardless of whether $\mathcal{J}$ and $\mathcal{K}$ are disjoint or not. If $\mathcal{J}\cap\mathcal{K}\neq\emptyset$, then (since $\mathcal{I}=\emptyset$) any element $a\in\mathcal{J}\cap\mathcal{K}$ must satisfy the conditions of Lemma 10.2 (i), so that

$$A\cap\Pi(a,p_i^2)=a*F_j*F_k.$$

It follows that the set $\mathcal{J}\setminus\mathcal{K}$ is M-fibered in the p_j direction, and $\mathcal{K}\setminus\mathcal{J}$ is M-fibered in the p_k direction.

We begin with the case when at least one of Φ_{M_j} and Φ_{M_k} divides A.

Lemma 13.2 ([25], Lemma 9.33)**.** *Assume (F3), and that $\Phi_{M_k}|A$. Then*

$$\Phi_{p_j^2}|A.$$

Furthermore, $\mathcal{K}$ is M_k-fibered in the p_j direction, so that for every $a_k\in\mathcal{K}$ we have

$$\mathbb{K}^{M_k}_{M_k/p_j}[a_k]=p_k\cdot\phi(p_j). \tag{13.2}$$

and

$$A\cap\Pi(a_k,p_i^2)\subset\Lambda(a_k,p_i^2p_j).$$

The same holds with p_k and p_j interchanged.

Lemma 13.3 ([25], Lemma 9.34)**.** *Assume (F3). The following holds true:*

(i) *If $\Phi_{p_i^2}\Phi_{M_k}|A$, then A is contained in a subgroup.*
(ii) *If $\Phi_{p_i}|A$, then $|A\cap\Pi(a,p_i)|=p_jp_k$ for all $a\in A$. Moreover, for every $a\in A$ we have either $A\cap\Pi(a,p_i)\subset\mathcal{J}$ or $A\cap\Pi(a,p_i)\subset\mathcal{K}$.*

Corollary 13.4. *Assume that (F3) holds, and that*

$$\Phi_{p_i}\Phi_{M_k}|A. \tag{13.3}$$

Then

(i) *$|A\cap\Pi(a,p_i^2)|=p_jp_k$ for every $a\in\mathcal{K}$.*
(ii) *Assume, in addition, that $\Phi_{M_j}|A$. Then the conditions of Theorem 5.2 (the slab reduction) are satisfied in the p_i direction, after interchanging A and B.*

Proof. Let $a\in\mathcal{K}$, then by (13.2) we have $|A\cap\Pi(a,p_i^2)|\geq p_jp_k$, and by Lemma 7.3 this must hold with equality. This implies (i).

For (ii), we have $\Phi_{p_i^2}|B$ by (13.3). Moreover, in this case (i) also holds for $a\in\mathcal{J}$, hence for all $a\in A$. This, however, means that (5.4) holds with A and B interchanged. By Corollary 5.7 (ii), the slab reduction conditions are satisfied as claimed. □

Lemma 13.5. *Assume (F3) and (13.3). Then $\Phi_{N_j}\nmid A$.*

Proof. Assume, by contradiction, that $\Phi_{N_j}|A$. By (13.1) and Corollary 10.8, we must have $p_j=\min_\nu p_\nu$. We first claim that

$$\text{if } \Phi_{N_j}|A, \text{ then } \mathcal{J} \text{ must be } N_j\text{-fibered on } D(N_j) \text{ grids.} \tag{13.4}$$

Indeed, suppose that (13.4) fails. When $p_j=2$, Lemma 10.9 applied to p_j implies $\Phi_{p_i^2}|A$, which contradicts the assumption (13.3). The rest of the argument, as well as the proof of (13.4) when $p_j>2$, appears in [25], Lemma 9.35. □

Assume next that $\Phi_{M_j} \nmid A$. Then, by Lemma 13.5,

$$\Phi_{N_j}\Phi_{M_j} | B. \tag{13.5}$$

Lemma 13.6. *Assume (F3), (13.3), and (13.5). Then B is N_j-fibered in the p_i direction, so that for every $b \in B$,*

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_j}[y] = 1 \textit{ for every } y \in \mathbb{Z}_M \textit{ with } (b-y, M) = M/p_i, \tag{13.6}$$

Proof. By (13.5) and Corollary 7.7 with $c_0 = 1$, B is a union of pairwise disjoint N_i-fibers in the p_j and p_k directions. It follows that for every $b \in B$, either (13.6) holds, or else

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_j}[y] = 1 \text{ for every } y \in \mathbb{Z}_M \text{ with } (b-y, M) = M/p_k. \tag{13.7}$$

Assume, by contradiction, that (13.7) holds for some $b \in B$. Since $M/p_k \in \mathrm{Div}(A)$, we must have $p_k < p_j$ and $M/p_jp_k \in \mathrm{Div}(B)$. On the other hand, $\mathcal{K}$ satisfies (13.2), so that $p_k < p_j$ implies $M/p_jp_k \in \mathrm{Div}(A)$, contradicting divisor exclusion. □

Corollary 13.7. *Assume (F3), (13.3), and (13.5). Then the conditions of Theorem 5.2 are satisfied in the p_i direction, after interchanging A and B.*

Proof. By Lemma 5.6 (III), it suffices to prove that for all $a \in A$, $b \in B$, and for all $y \in \mathbb{Z}_M$ with $(y-b, M) = M/p_i$, we have

$$B_{y,a} \subset \Pi(y, p_i^2). \tag{13.8}$$

For $a \in \mathcal{K}$, we have $A \cap \Pi(a, p_i) = A \cap \Pi(a, p_i^2)$ by Lemma 13.3 (ii) and Lemma 13.4 (i). This clearly implies (13.8).

Assume now that $a \in \mathcal{J}$. If $y \in B$, (13.8) holds trivially. Otherwise, we have $\mathbb{B}_{M/p_j}[y] = 1$ by (13.6), so that

$$\langle \mathbb{A}[a], \mathbb{B}[y] \rangle = \frac{1}{\phi(p_j)} \mathbb{A}_{M/p_j}[a] \mathbb{B}_{M/p_j}[y] = 1,$$

which proves (13.8). □

It remains to prove Proposition 13.1 under the assumption that

$$\Phi_{M_\nu} \nmid A \text{ for } \nu \in \{j, k\}. \tag{13.9}$$

Without loss of generality, we may also assume that

$$p_k > p_j. \tag{13.10}$$

By Corollary 10.8, it follows that $\Phi_{N_k}\Phi_{M_k} | B$. As in the proof of Lemma 13.6, we see that B mod N_k is a union of pairwise disjoint N_k-fibers in the p_i and p_j directions, so that every $b \in B$ satisfies at least one of the following:

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_k}[y] = 1 \text{ for every } y \in \mathbb{Z}_M \text{ with } (b-y, M) = M/p_i. \tag{13.11}$$

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_k}[y] = 1 \text{ for every } y \in \mathbb{Z}_M \text{ with } (b-y, M) = M/p_j, \tag{13.12}$$

In particular,

$$\{M/p_i, M/p_j, M/p_ip_k, M/p_jp_k\} \cap \mathrm{Div}(B) \neq \emptyset. \tag{13.13}$$

Lemma 13.8. *Assume (F3), (13.9), and (13.10). Then:*

(i) $\Phi_{N_j} \nmid A$,
(ii) *B is N_j-fibered in the p_i direction,so that (13.6)holds for all $b \in B$,*

(iii) $\Phi_{p_i}|A$.

Proof. We start with (i). Assume for contradiction that $\Phi_{N_j}|A$.

We first claim that $\mathcal{J}\setminus\mathcal{K}$ must be N_j-fibered in the p_j direction. Indeed, if M is odd, this follows from Lemma 10.7 applied to p_j, (13.13), and Lemma 10.5 (i). If M is even but $p_j > 2$, this follows from Corollary 10.8 and, again, Lemma 10.5 (i),

Assume now that $p_j = 2$. Then the conclusion of Lemma 10.9 (i) cannot hold for p_j, since (13.13) contradicts (10.11). Therefore the conclusion of Lemma 10.9 (ii) holds, so that $\mathcal{J}\setminus\mathcal{K}$ modulo N_j contains diagonal boxes and $\{M/p_i, M/p_j, M/p_jp_k\} \subset \mathrm{Div}(A)$. In addition, it follows from (13.13) that (13.12) cannot hold for any $b \in B$. Hence B is N_k-fibered in the p_i direction, with

$$\mathbb{B}^{N_k}_{N_k/p_i}[b] = \mathbb{B}_{M/p_ip_k}[b] = \phi(p_i) \text{ for all } b \in B.$$

This means that B satisfies the assumptions of Lemma 7.2, with $m = M/p_ip_k$ and $s = p_i^2$. Hence $\Phi_{p_i^2}|B$. On the other hand, Lemma 10.9 implies $\Phi_{p_i^2}|A$, which is a contradiction. By Lemma 10.5 (i), the set $\mathcal{J}\setminus\mathcal{K}$ must be N_j-fibered in the p_j direction.

Let $a_j \in \mathcal{J}\setminus\mathcal{K}$. We now consider two cases.

- Suppose that $\Phi_{p_i}|A$. By Lemma 13.3 (ii), we have $A\cap\Pi(a_j,p_i) \subset \mathcal{J}$ and $|A\cap\Pi(a_j,p_i)| = p_jp_k$. If there was an element $a \in (\mathcal{J}\cap\mathcal{K})\cap\Pi(a_j,p_i)$, it would follow that $A\cap\Pi(a_j,p_i) = a * F_j * F_k$, and in particular $a_j \in a * F_j * F_k$, contradicting the choice of a_j. Therefore $A\cap\Pi(a_j,p_i) \subset \mathcal{J}\setminus\mathcal{K}$. But then the fibering of $\mathcal{J}\setminus\mathcal{K}$ implies that p_jp_k is divisible by p_j^2, a contradiction.
- Assume now that $\Phi_{p_i^2}|A$. Let A' be a translate of A such that $a_j \in A'_{p_i}$. By the cyclotomic divisibility assumption, we have $|A'_{p_i}| = p_jp_k$. On the other hand, by the fibering properties of A,

  $$p_jp_k = |A'_{p_i}| = c_jp_j^2 + c_kp_k, \quad c_j > 0.$$

  Thus $c_j = p_kc'_j$ and $p_j = c'_jp_j^2 + c_k$ with $c'_j > 0$, a contradiction.

Therefore $\Phi_{N_j} \nmid A$, proving (i).

Next, we prove (ii). By (i) together with (13.9), we have $\Phi_{N_j}\Phi_{M_j}|B$. As in the proof of Lemma 13.6, B mod N_j is a union of pairwise disjoint N_j-fibers in the p_i and p_k directions, so that every element of B satisfies at least one of (13.6) and (13.7). However, by (13.10), we must in fact we have (13.6) for all $b \in B$, otherwise $M/p_k \in \mathrm{Div}(A)\cap\mathrm{Div}(B)$ which is a contradiction. This proves (ii).

Finally, the N_j-fibering in (ii) implies that the assumptions of Lemma 7.2 hold for B, with $m = M/p_ip_j$ and $s = p_i^2$. It follows that $\Phi_{p_i^2}|B$, and therefore $\Phi_{p_i}|A$. □

Lemma 13.9. *Assume (F3), (13.9), and (13.10). Then the conditions of Theorem 5.2 (the slab reduction) are satisfied in the p_i direction, after interchanging A and B.*

Proof. We verify that the condition (II) of Lemma 5.6 holds. That is, given $r \in R$, we show that every M-fiber in the p_i direction splits with parity (rA, B). Since the only property of the set A that our proof uses is the fact that every element belongs to an M-fiber in either the p_j or the p_k direction, and this property is preserved under the mapping $A \mapsto rA$, it suffices to consider the case $r = 1$.

Let $x \in \mathbb{Z}_M$, and let $a \in A, b \in B$ satisfy $x = a + b$. If $a \in \mathcal{J}$, then $x * F_i * F_j$ is tiled by $a * F_j \subset A$ and the N_j-fiber in the p_i direction in B containing b, provided by (13.6).

Suppose now that $a \in \mathcal{K}$. If b belongs to an N_k-fiber in the p_i direction in B (as in (13.11)), then $x * F_i * F_k$ is tiled by $a * F_k \subset A$ and that fiber (this is the same argument as for $a \in \mathcal{J}$, with j and k interchanged). If on the other hand b satisfies (13.12), with each element b' of its N_k-fiber in the p_j direction belonging to a N_j-fiber in the p_i direction, then $\Lambda(x, D(M))$ is tiled by $a * F_k \subset A$ and all $p_i p_j$ elements $b'' \in B$ satisfying $(b - b'', M) \in \{M, M/p_i, M/p_i p_j, M/p_j p_k\}$. In both cases, $x * F_i$ splits with parity (A, B), as required. □

This concludes the proof of Proposition 13.1.

14. Acknowledgements

The first author is supported by NSERC Discovery Grant 22R80520. The second author is supported by ISF Grant 607/21.

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