Matching, coupling, point processes: exercises

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Ratings: **.** routine (but good practice); ***** more interesting or challenging;
****** research problem (let me know if you have ideas)

Point processes

1. If Π is a translation-invariant simple point process, show that a.s., $\Pi(\mathbb{R}^d)$ equals 0 or ∞.

2* Show that

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\Big[a > |x_1| > |x_1 - x_2| > \cdots > |x_{n-1} - x_n|\Big]\, dx_1 \cdots dx_n = \frac{(\omega_d a^d)^n}{n!},$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. (This was used in the proof that the Poisson process has no descending chains).

3. Show that for a simple point process Π with n-th joint intensity $\mu^{(n)}$, for any Borel $S \subset \mathbb{R}^d$,

$$\mu^{(n)}(S^n) = \mathbb{E}\Big[N(N-1)\cdots(N-n+1)\Big],$$

where $N = \Pi(S)$. Also check this directly in the case of a Poisson process.

4. Show that a homogeneous Poisson process Π on $\mathbb{R}^d$ is a.s. non-equidistant; that is, there do not exist Π-points w, x, y, z with $\{w, x\} \neq \{y, z\}$ and $|w-x| \neq |y-z| > 0$.

5. Give an example of a locally finite, ergodic, translation-invariant simple point process with infinite intensity. (Recall that locally finite means that any bounded set contains only finitely many points a.s.).

6* Give an example of a translation-invariant point process of finite intensity that has infinite descending chains.

Palm processes

7. In a discrete setting the Palm process can be constructed by elementary conditioning. Let η be a translation-invariant random element of $\{0,1\}^{\mathbb{Z}^d}$ with law $\mathbb{P}$ and $\mathbb{E}\eta(0) =: p \in (0,1)$. Define η^*, with law $\mathbb{P}^*$, by $\mathbb{P}^*(\eta^* \in \cdot) = \mathbb{P}(\eta \in \cdot \mid \eta(0) = 1)$. Let $A \subseteq \{0,1\}^{\mathbb{Z}^d}$ be an event and let $S \subset \mathbb{Z}^d$ be finite. Show that

$$\mathbb{E} \sum_{x \in S:\, \eta(x)=1} \mathbf{1}[\theta^{-x}\eta \in A] = p \cdot \#S \cdot \mathbb{P}^*(\eta^* \in A).$$

Also state and prove an extension to the case where the indicator is replaced with $f(\theta^{-x}(\eta,\gamma),x)$, where γ is a jointly translation-invariant "background" process, and $f \geq 0$.

8. As usual let $\mathcal{M}$ be a translation-invariant matching scheme of jointly ergodic $\mathcal{R}, \mathcal{B}$ of equal intensities, let $(\mathcal{R}^*, \mathcal{B}^*, \mathcal{M}^*)$ be the Palm version of $(\mathcal{R}, \mathcal{B}, \mathcal{M})$ in which we condition on a red point at O, and let $X = |\mathcal{M}^*(0)|$. Suppose instead that $(\mathcal{R}^\circ, \mathcal{B}^\circ, \mathcal{M}^\circ)$ is the Palm version in which we condition on a *blue* point at O. Show that $X \stackrel{d}{=} |\mathcal{M}^\circ(0)|$.

2-color matching

9. Let $\mathcal{M}$ be a translation-invariant matching of two independent Poisson processes $\mathcal{R}, \mathcal{B}$ of intensity 1. Prove that $\mathbb{P}^*(X > r) > e^{-cr^d}$ for some $c = c(d) > 0$.

10* Let μ be a probability measure on the positive integers. Starting with a Poisson process of intensity 1 on $\mathbb{R}^d$, put a random number of red particles at each Poisson point, where the numbers are i.i.d. with law μ (thus the red particles form a compound (non-simple) point process). Generate blue particles in the same way using an independent Poisson process and another probability measure ν. If μ and ν both have mean ∞, prove that there is a translation-invariant perfect matching scheme of red particles to blue particles. (Hint: use additional randomness to split the particles into a suitable countable sequence of simple point processes, and use a matching scheme we already have).

11. For X a positive random variable and $\alpha \in (0,1)$, prove that $\mathbb{E}X^\alpha < \infty$ implies $\mathbb{E}(X \wedge t) = o(t^{1-\alpha})$ as $t \to \infty$. (This was used in the lower bound for 2-color matching in dimension 1).

12. Show that if $X_n \xrightarrow{L^1} c$ (where c is a constant) and $X_n \stackrel{d}{=} Y_n$, then $Y_n \xrightarrow{L^1} c$. (This was used in the lower bound for 2-color matching in dimension 2).

13. For a translation-invariant matching scheme in $d = 1$, show that
$\mathbb{E}\#\{\text{edges crossing } O\} \leq \mathbb{E}^* X$.

14* Isometry-invariance can make a difference. Take a Poisson process of intensity 2 on $\mathbb{R}$, and color the points *alternately* red and blue, flipping a fair coin to decide on the color of the point closest to 0. This gives (dependent) jointly translation-invariant and ergodic point processes $\mathcal{R}$ and $\mathcal{B}$. Show that there is a translation-invariant matching of $\mathcal{R}$ and $\mathcal{B}$ in which X has exponential tails, but any translation-ergodic, isometry-invariant matching has $\mathbb{E}^* X = \infty$.

15. If U and V are independent Poisson random variables with common mean μ, show that

$$\mathbb{P}\Big(\frac{U-V}{U+V} > a\Big) \leq \exp -\mu(a^2 + O(a^4))$$

uniformly in μ as $a \downarrow 0$. (This is an ingredient in the exponential tail bound for the allocation in $d \geq 3$.)

16** Find a simple explicit matching scheme with exponential tails for two independent Poisson processes in $d \geq 3$. Ideally, a factor matching and $\mathbb{P}^*(X > r) < e^{-cr^d}$.

Stable matching

17. Show that in a stable two-color matching M in $d = 1$, edges cannot "cross", i.e. there do not exist points $w < x < y < z$ with $(w, y), (x, z) \in M$.

18* Find disjoint sets $R, B \subset \mathbb{R}$, with $R \cup B$ countable and non-equidistant, such that:
(a) there is no stable partial matching of R to B;
(b) $R \cup B$ is discrete, and there is more than one stable perfect matching.
What does the iterated mutually nearest neighbor matching algorithm do in these cases?

19* In $\mathbb{R}$, suppose that a red-blue pair may be matched only if the red point lies to the left of the blue point, and that points prefer closer partners, provided they are on the correct side (left or right). Define one-sided-stable matching accordingly. Show that, if the red and blue points form a locally finite non-equidistant set, there is a unique one-sided-stable matching, and the partner of a red point x is the blue point located at

$$\inf \big\{y > x : \mathcal{R}([x, y]) = \mathcal{B}([x, y])\big\}.$$

Deduce that in the case of independent Poisson processes, X can be expressed exactly in terms of a random walk.

20* For the alternating-color processes in Exercise 14, show that the stable matching has $\mathbb{P}^*(X > r) < c/r$ for some c and all $r > 0$.

21. For any non-equidistant discrete set of points in $\mathbb{R}^d$ with no descending chains, show that there is a unique one-color stable partial matching. In the case of a translation-invariant point process, show that it is a.s. a perfect matching.

22* For *any* translation-invariant non-equidistant simple point process on $\mathbb{R}^d$ with intensity 1 and no descending chains, prove that for the one-color stable matching, $\mathbb{P}^*(X > r) < c/r^d$ for all $r > 0$, where c depends only on d. (Hint: how close can two r-bad points be?)

23. Show that, if the simple point process Π is insertion-tolerant, then so is $\Pi + \delta_U$, where U is uniformly distributed on any given set of finite volume. (Hint: show first that $\Pi + \delta_x$ is insertion-tolerant for almost every deterministic x.)

24. Give examples of translation-invariant simple point processes of finite intensity with the following properties:
(a) insertion-tolerant but not deletion-tolerant;
(b) deletion-tolerant but not insertion-tolerant;
(c) deletion-tolerant but $\Pi^* \not\prec \Pi + \delta_0$.

25** Improve the tail bounds for the stable matching of two independent Poisson processes in $d \geq 2$. Is the correct power $d/2$?

Further topics

26** Let $\mathcal{R}, \mathcal{B}$ be independent Poisson processes of intensity 1 in $\mathbb{R}^2$. Does there exist a translation-invariant matching scheme in which the line segments joining matched pairs do not cross?

27** In the setting of Question 26, does there exist a minimal matching?

28* In the stable allocation of Poisson to Lebesgue, prove that a.s. only finitely many territories intersect $B(0,1)$. Deduce that each territory has only finitely many connected components.