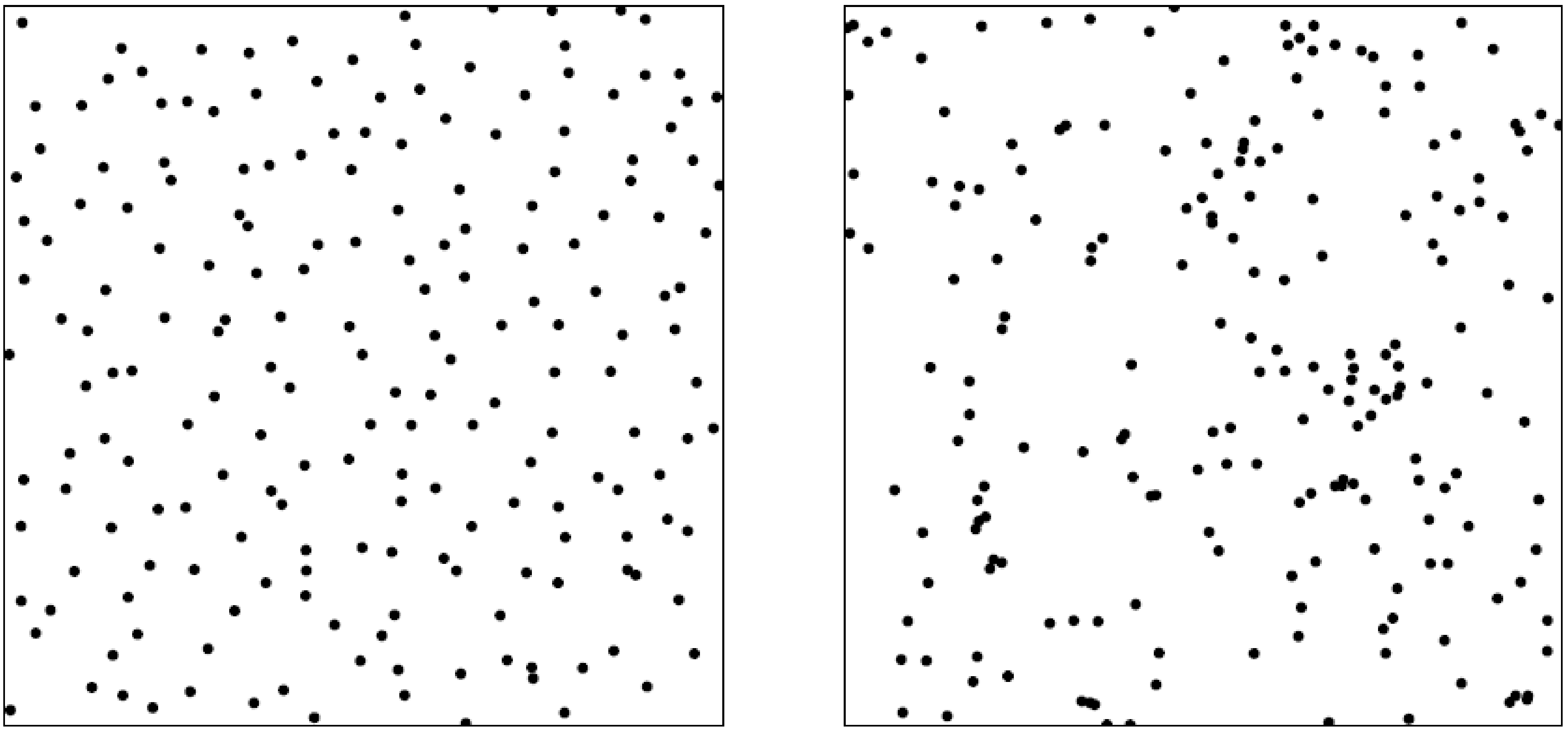
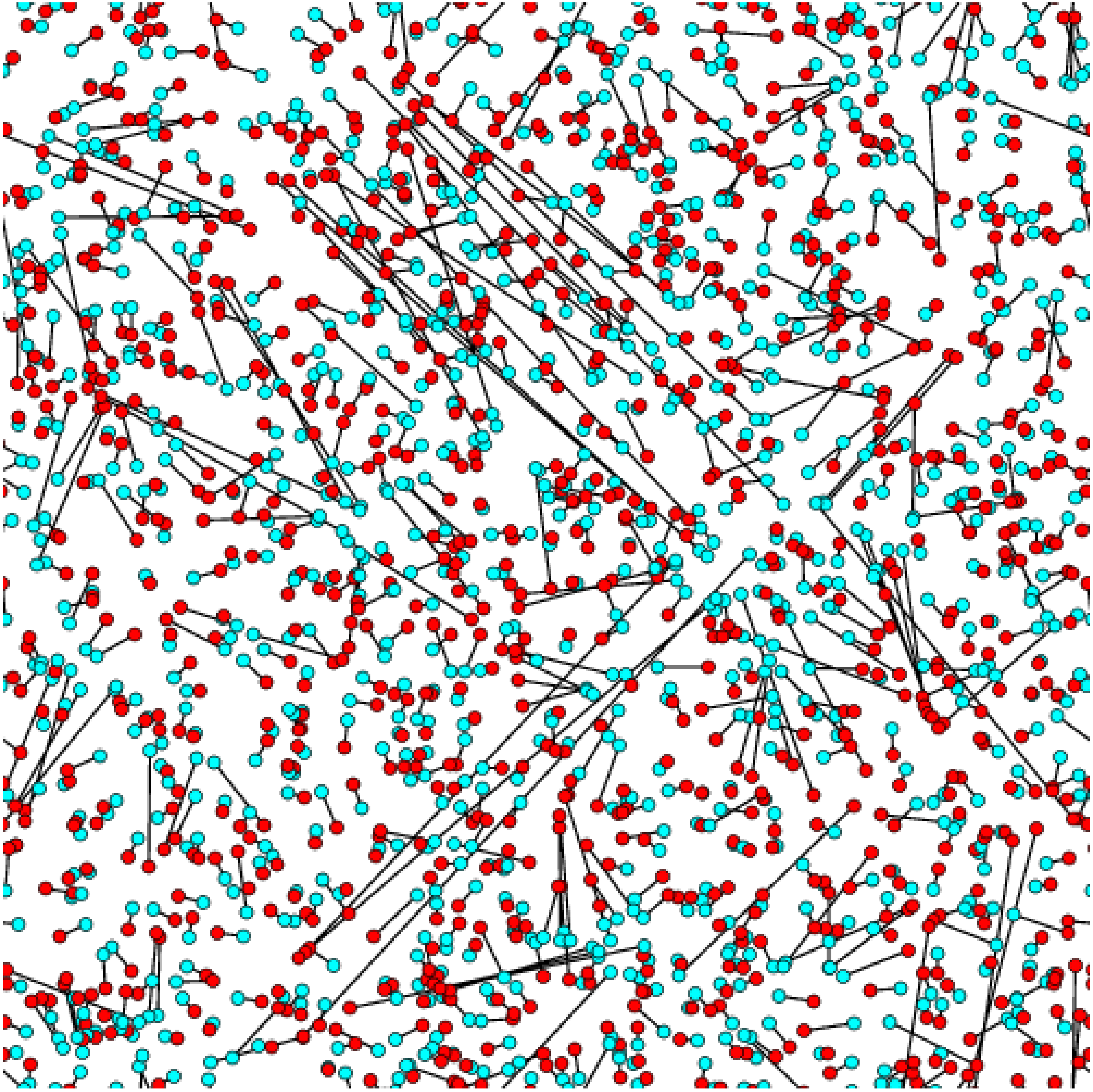
Notes on Matching and Point Processes

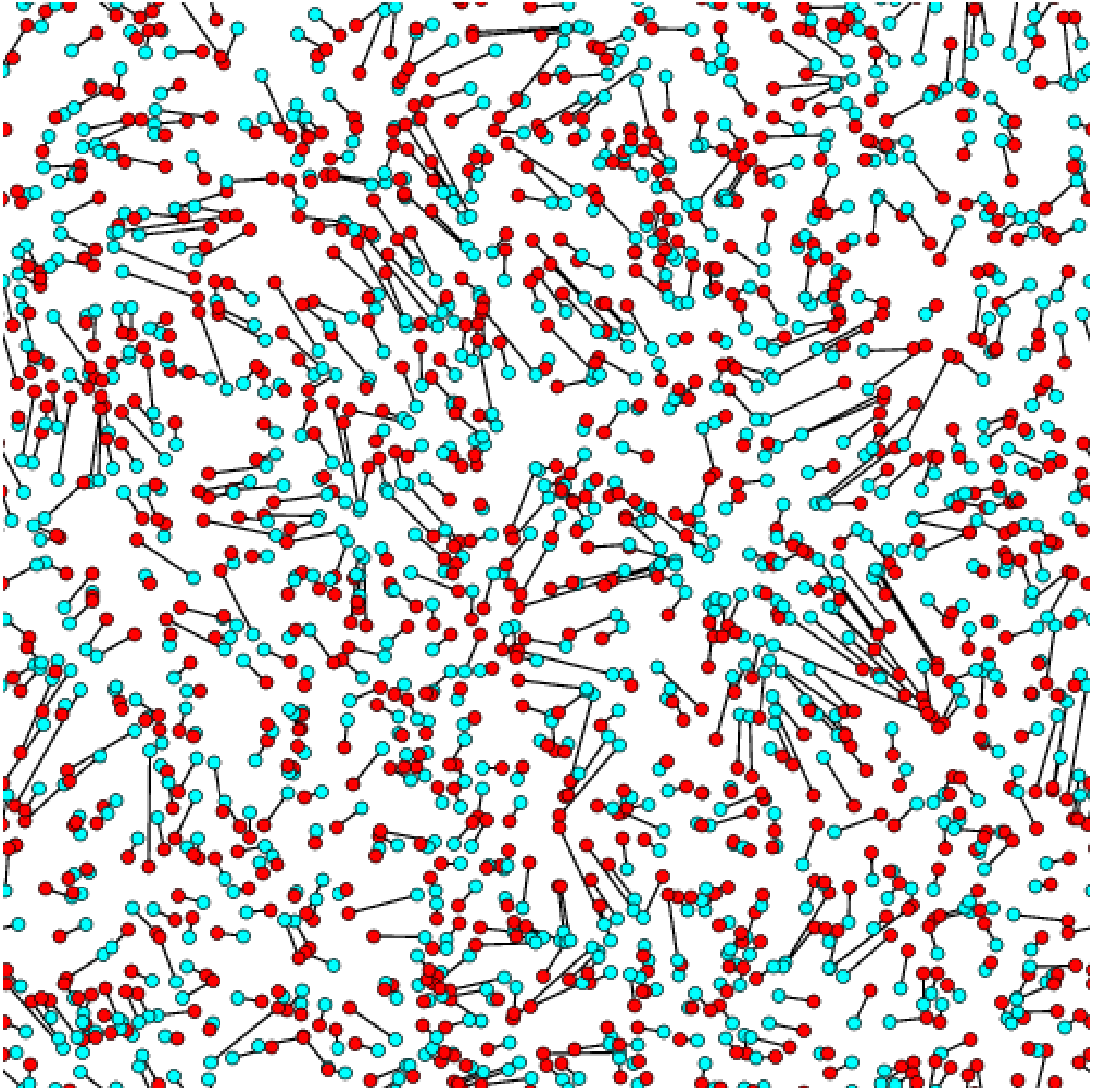
Cornell, 2009

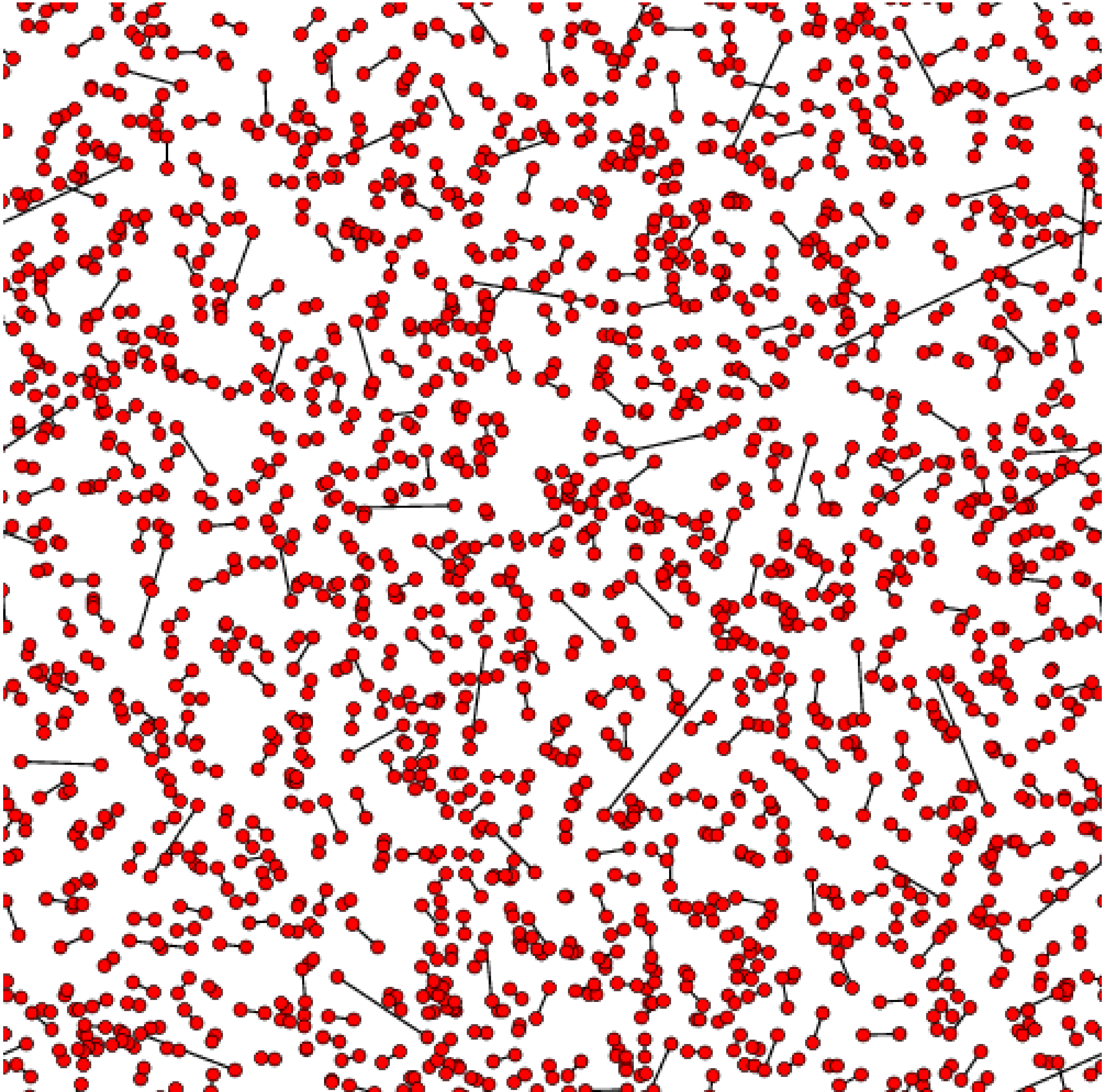
Pictures:

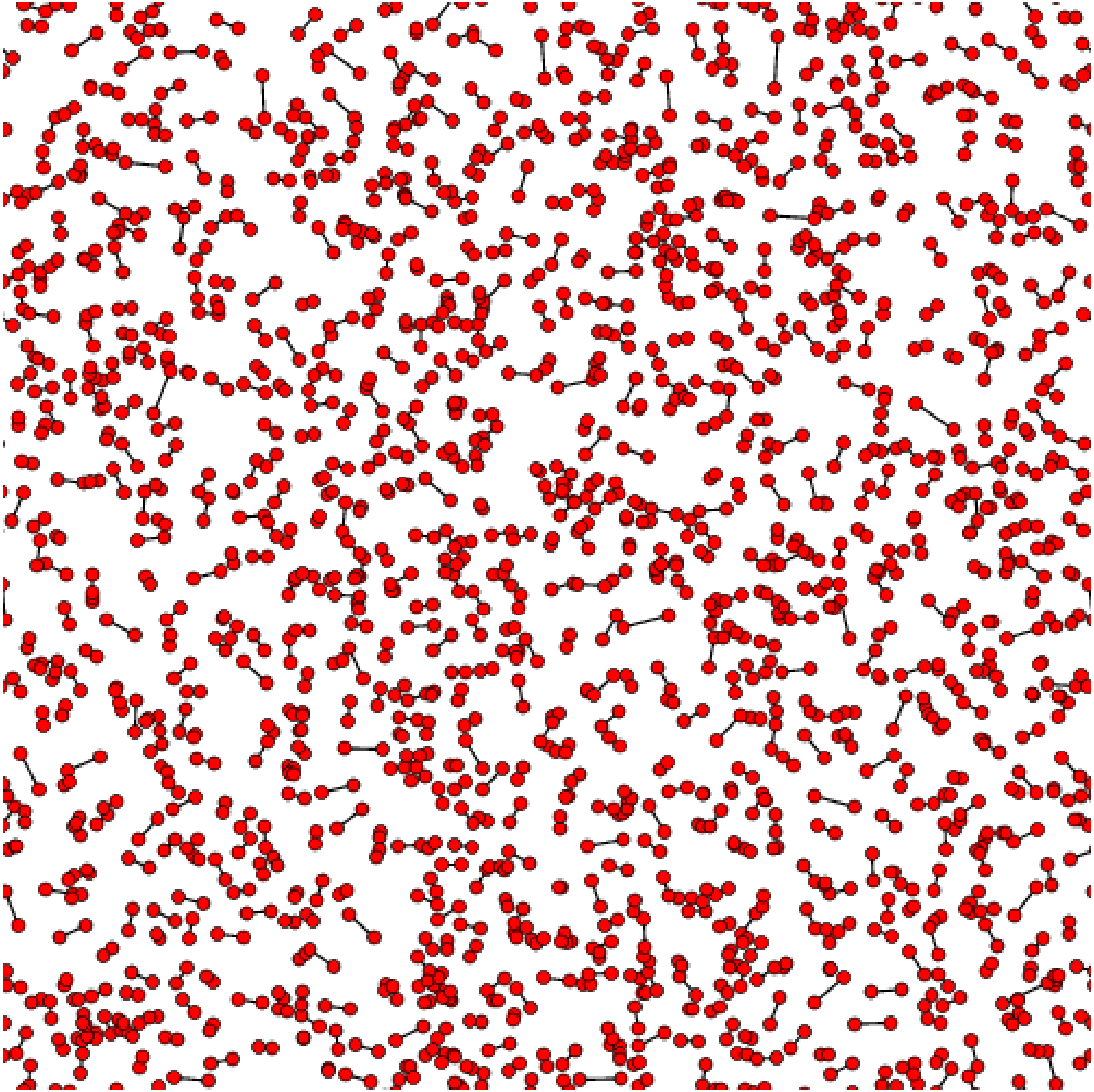
- Zeros of Gaussian analytic function / Poisson process
- 2-color stable matching
- 2-color minimum matching
- 1-color stable matching
- 1-color minimum matching
- stable allocation

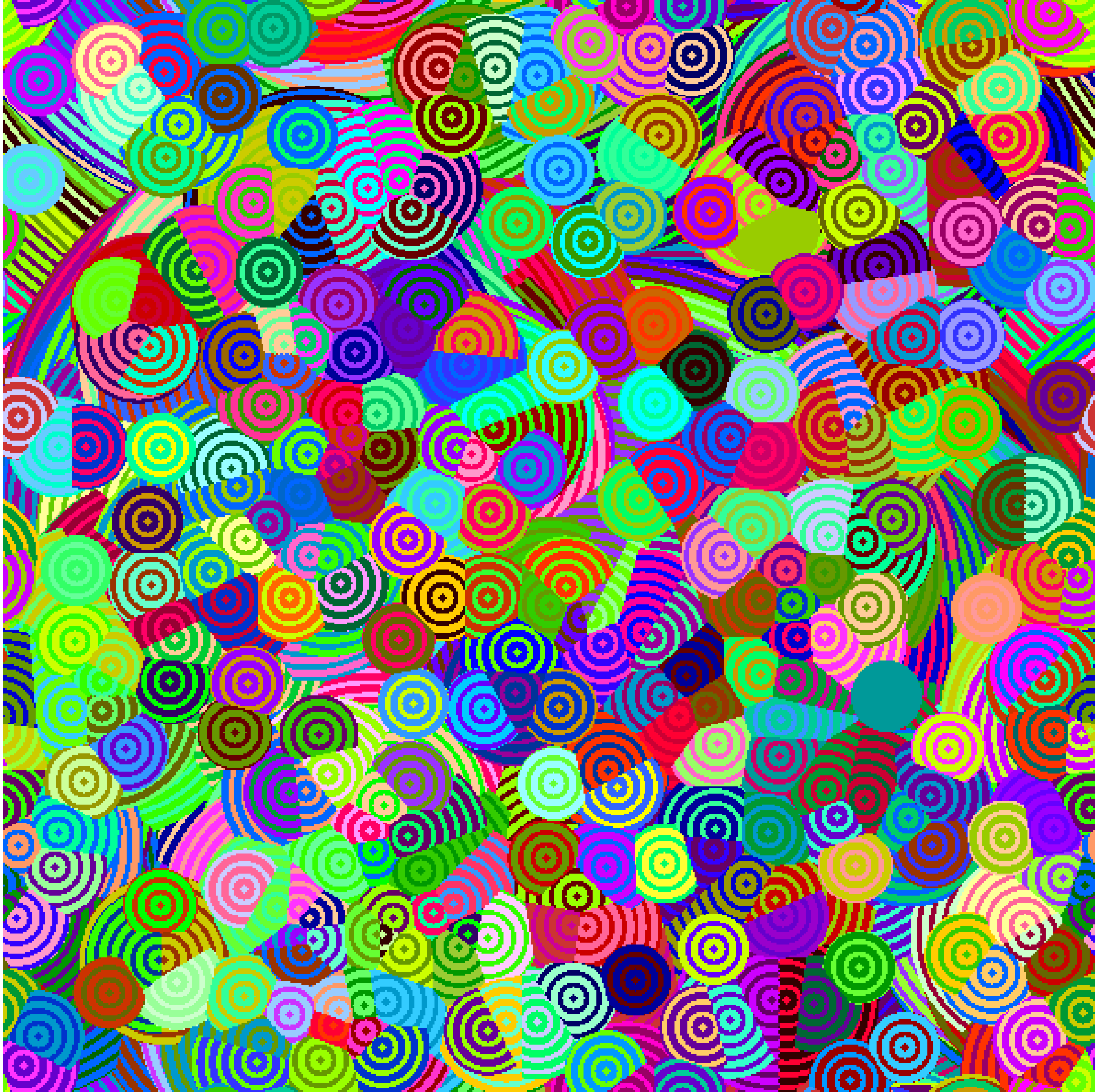












Exponential Matching

Theorem For 2 independent Poisson processes in $d \geq 3$, $\exists$ a translation-invariant matching scheme with

$$P^*(X > r) < \exp[-c\, r^d]$$

Several approaches to proving this and related results:

1. Theorem (Talagrand, 1994)
For n red, n blue points, uniformly random in $[0,n^{1/d})^d$, $d \geq 3$, $\exists$ a matching with

$$P\left(n^{-1} \sum_{x \text{ red}} e^{c|x-\mathcal{M}(x)|^d} \leq 2\right) \geq 1 - n^{-2}$$

Proof: difficult, abstract.

Can deduce Poisson matching result: tile space with copies of finite matching, conditioned on good event, randomize position of O, take limit in distribution as n -> ∞.

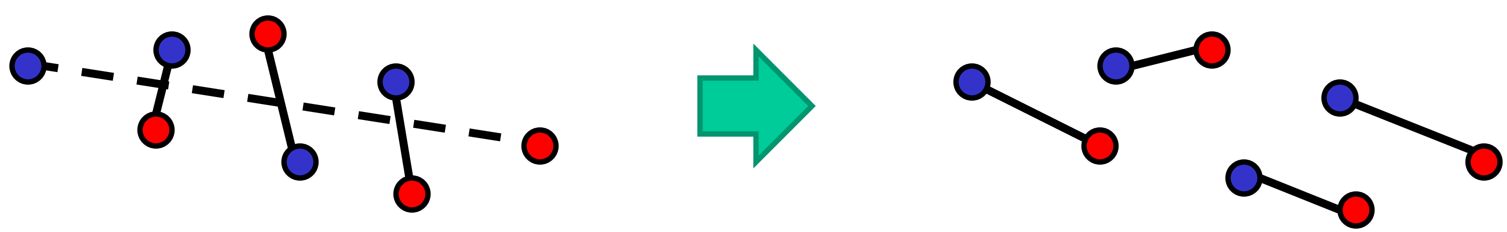
2. Related problem: i.i.d. coin flips on Z^d, $P(+) = P(-) = \frac{1}{2}$.

Theorem (Timar, preprint) For $d \geq 3$, $\exists$ a translation invariant matching with

$$P^*(X > r) < \exp -cr^{d-2}$$

(in fact, matching is a deterministic function of the coin flips)

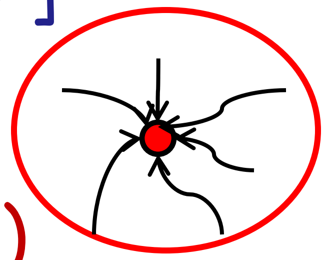
Proof difficult, uses rematching:



3. Related problem: allocate volume 1 to each point of a Poisson process, forming a partition.

Theorem (Chaterjee, Peled, Peres, Romik, to appear). For d ≥ 3, *gravitational allocation* gives

$$P[\mathrm{diam}(\mathrm{cell}(O)) > r] < \exp[-c\, r\, (\log r)^{a}]$$

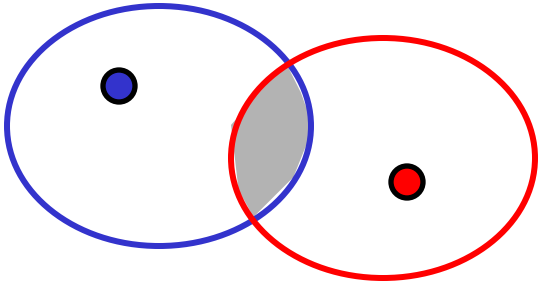


(Cell = basin of attraction of point for a intertialess particle under Newtonian gravity)

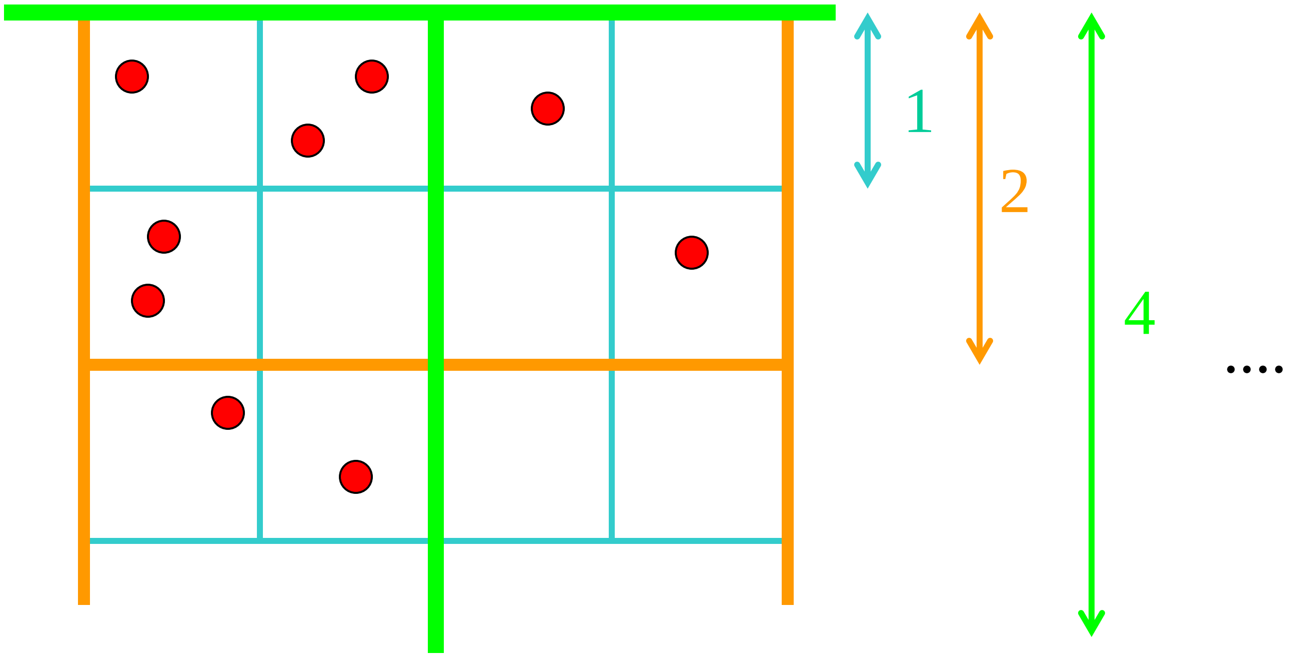
Proof difficult.

Given this, can get a red-blue "fractional matching":

$$\mathrm{wt}[(r,b)] = \mathcal{L}[C_R(r) \cap C_R(b)]$$



4. Another allocation with exponential tails (after Ajtai, Komlos,Tusnday; Talagrand, Yukich).



Invariant dyadic partition

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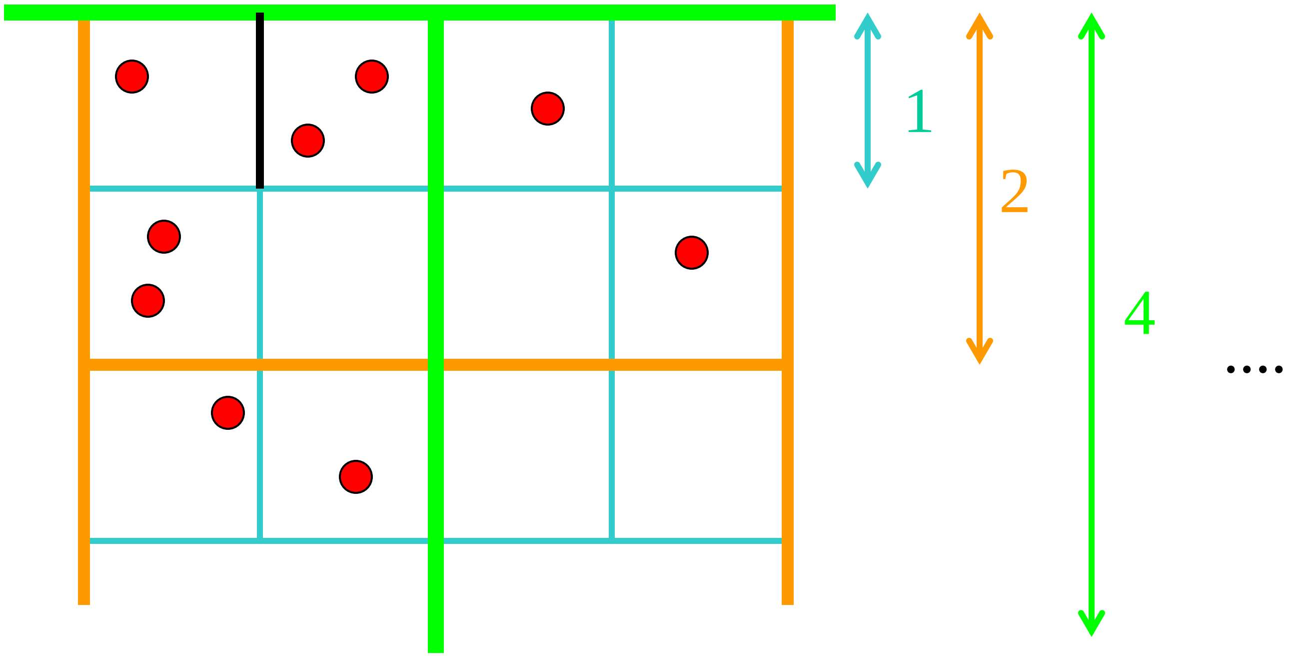

Repartition to equalize points per unit volume

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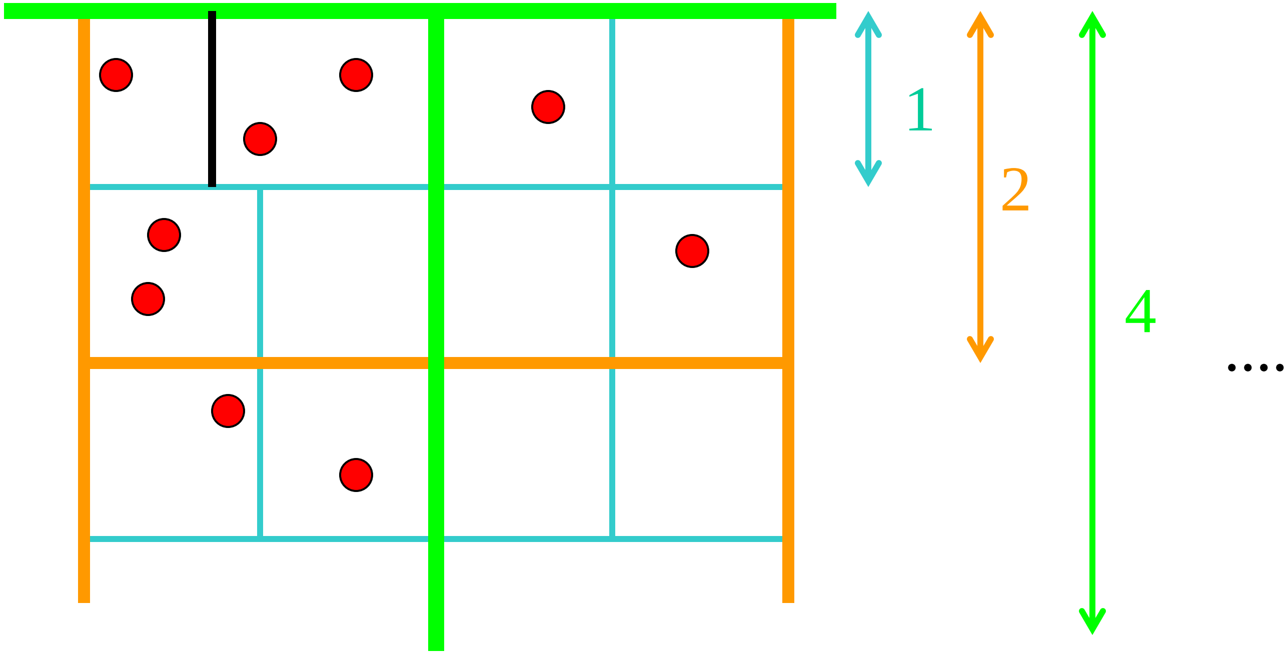

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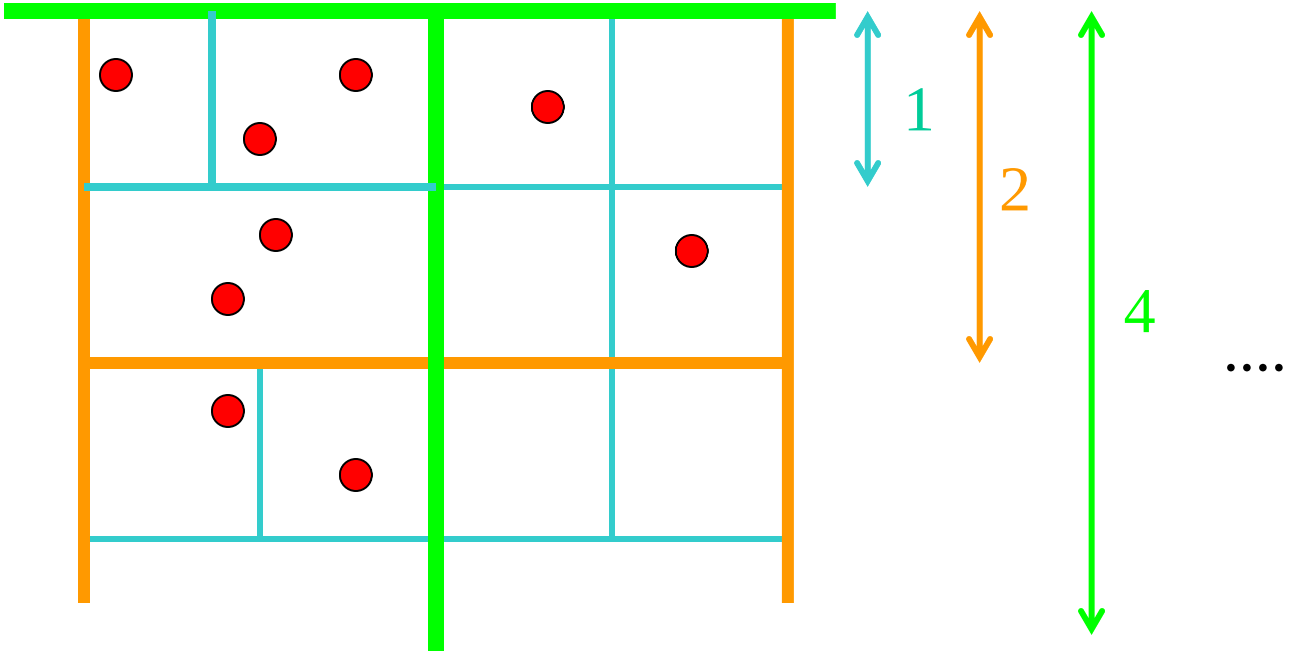

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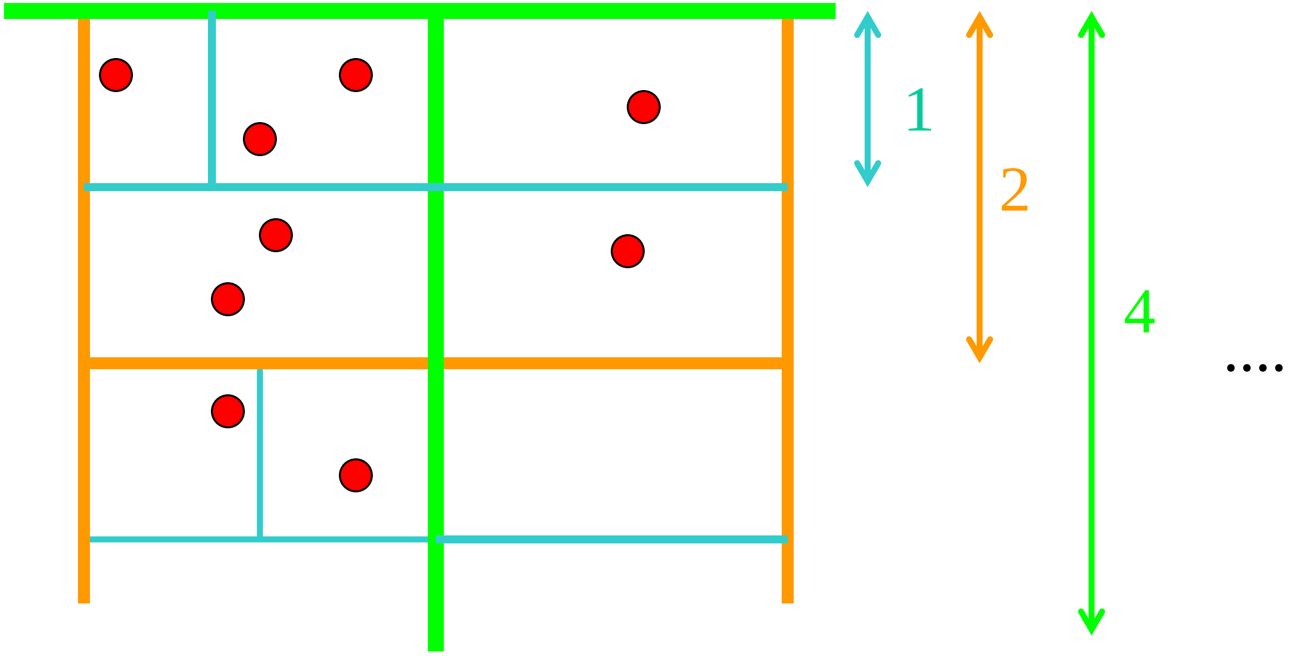

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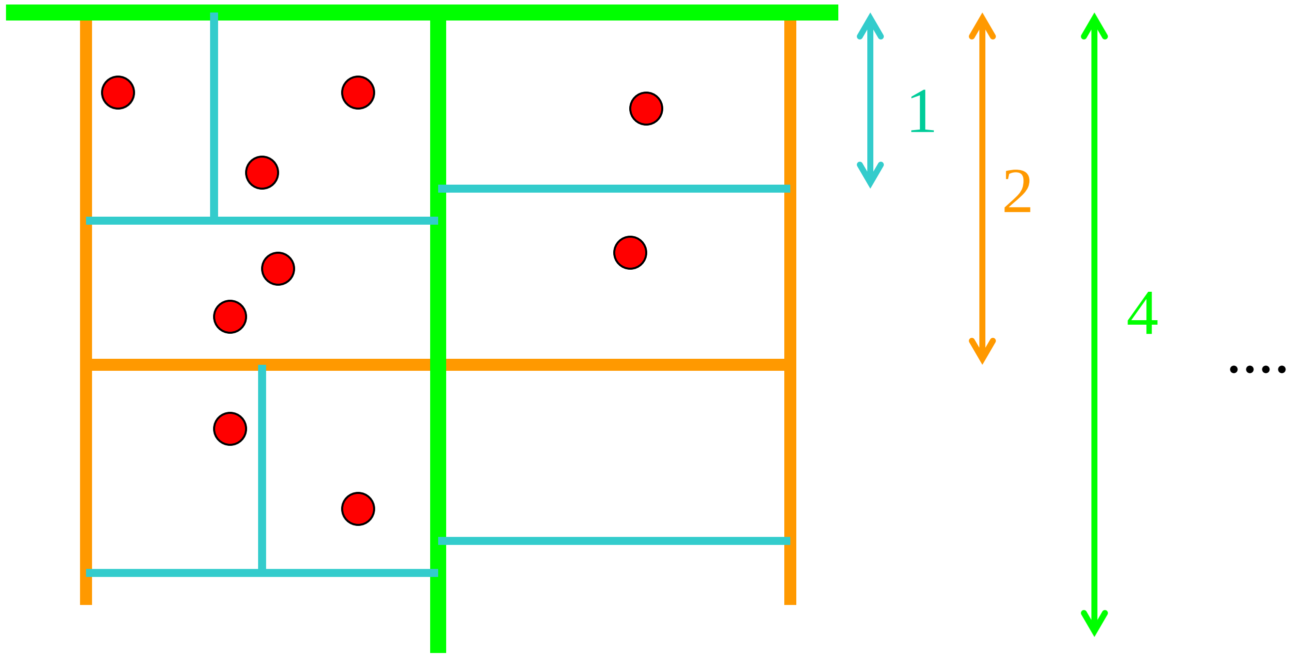

Repartition to equalize points per unit volume

Iterate...

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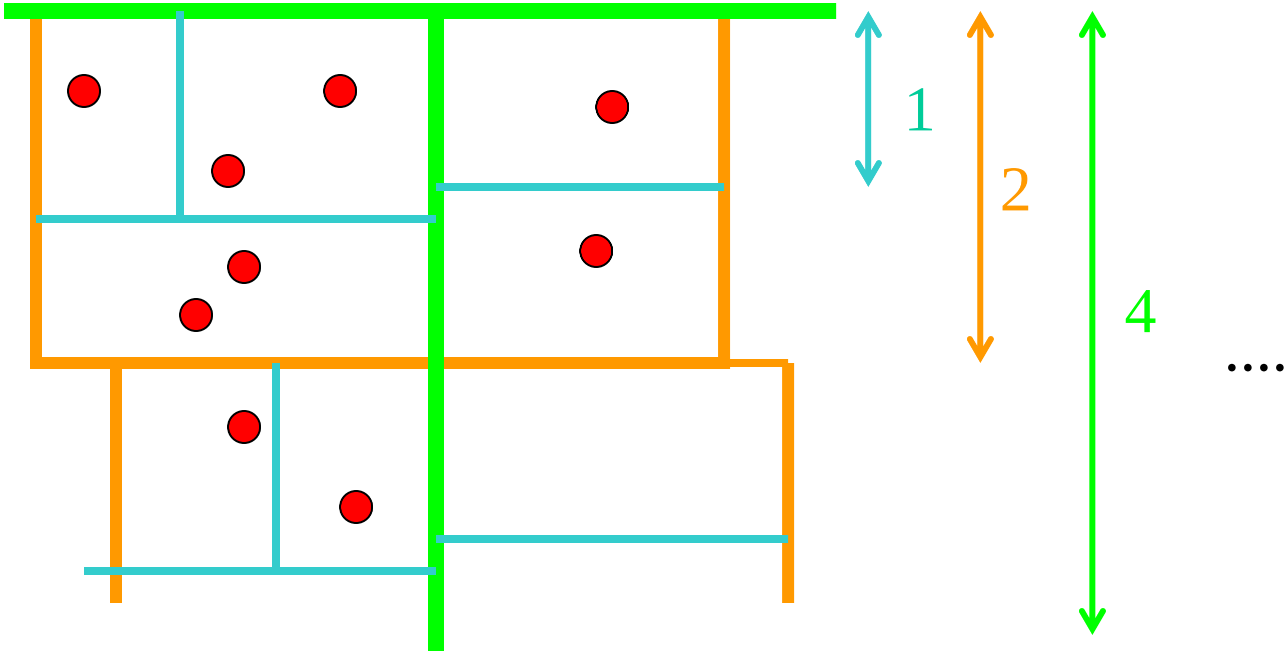


Repartition to equalize points per unit volume

Iterate...

Get allocation of 1 unit volume to each point.

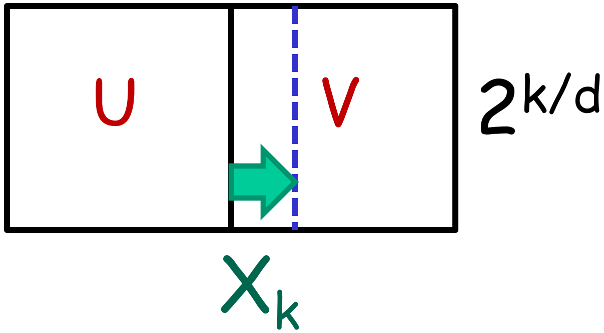
Max distance moved by sites in Q ≤ $c\sum_{k=1}^{\infty} X_k$

where $X_k = \frac{|U-V|}{U+V} 2^{k/d}$ U,V indep Poi(2^k)

typically: $U-V \approx 2^{k/2}$

$U+V \approx 2^k$

$X_k \approx 2^{-k(1/2-1/d)}$



large deviations: $P(X_k > a\,C^{-k}) < e^{-a^2}$, some $C > 1$

$P(\sum X_k > r) < e^{-cr^2}$

Get invariant allocation with

$P[\mathrm{diam(cell)} > r] < \exp[-cr^2]$

Challenge: find a *simple, explicit* red-blue matching with exponential tails in $d \geq 3$.