45. *(n-colorings of graphs)* A finite *graph* $\mathcal{G}$ of size N is a set of vertices $i \in \{1, 2, \ldots, N\}$ and a collection of edges (i, j) connecting vertex i with vertex j. An *n-coloring* of $\mathcal{G}$ is an assignment of one of n colors to each vertex in such a way that vertices connected by an edge have distinct colors. Let F be any field containing at least n elements. If we introduce a variable x_i for each vertex i and represent the n colors by choosing a set S of n distinct elements from F, then an n-coloring of $\mathcal{G}$ is equivalent to assigning a value $x_i = \alpha_i$ for each $i = 1, 2, \ldots, N$ where $\alpha_i \in S$ and $\alpha_i \neq \alpha_j$ if (i, j) is an edge in $\mathcal{G}$. If $f(x) = \prod_{\alpha \in S}(x - \alpha)$ is the polynomial in $F[x]$ of degree n whose roots are the elements in S, then $x_i = \alpha_i$ for some $\alpha_i \in S$ is equivalent to the statement that x_i is a solution to the equation $f(x_i) = 0$. The statement $\alpha_i \neq \alpha_j$ is then the statement that $f(x_i) = f(x_j)$ but $x_i \neq x_j$, so x_i and x_j satisfy the equation $g(x_i, x_j) = 0$, where $g(x_i, x_j)$ is the polynomial $(f(x_i) - f(x_j))/(x_i - x_j)$ in $F[x_i, x_j]$. It follows that finding an *n-coloring* of $\mathcal{G}$ is equivalent to solving the system of equations

$$\begin{cases} f(x_i) = 0, & \text{for } i = 1, 2, \ldots, N, \\ g(x_i, x_j) = 0, & \text{for all edges } (i, j) \text{ in } \mathcal{G} \end{cases}$$

(note also we may use any polynomial g satisfying $\alpha_i \neq \alpha_j$ if $g(\alpha_i, \alpha_j) = 0$). It follows by "Hilbert's Nullstellensatz" (cf. Corollary 33 in Section 15.3) that this system of equations has a solution, hence $\mathcal{G}$ has an n-coloring, unless the ideal I in $F[x_1, x_2, \ldots, x_N]$ generated by the polynomials $f(x_i)$ for $i = 1, 2, \ldots, N$, together with the polynomials $g(x_i, x_j)$ for all the edges (i, j) in the graph $\mathcal{G}$, is not a proper ideal. This in turn is equivalent to the statement that the reduced Gröbner basis for I (with respect to any monomial ordering) is simply $\{1\}$. Further, when an n-coloring does exist, solving this system of equations as in the examples following Proposition 29 provides an explicit coloring for $\mathcal{G}$.

There are many possible choices of field F and set S. For example, use any field F containing a set S of distinct n^{th} roots of unity, in which case $f(x) = x^n - 1$ and we may take $g(x_i, x_j) = (x_i^n - x_j^n)/(x_i - x_j) = x_i^{n-1} + x_i^{n-2}x_j + \cdots + x_i x_j^{n-2} + x_j^{n-1}$, or use any subset S of $F = \mathbb{F}_p$ with a prime $p \geq n$ (in the special case $n = p$, then, by Fermat's Little Theorem, we have $f(x) = x^p - x$ and $g(x_i, x_j) = (x_i - x_j)^{p-1} - 1$).

(a) Consider a possible 3-coloring of the graph $\mathcal{G}$ with eight vertices and 14 edges $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 4)$, $(2, 7)$, $(2, 8)$, $(3, 4)$, $(3, 6)$, $(3, 8)$, $(4, 5)$, $(5, 6)$, $(6, 7)$, $(6, 8)$, $(7, 8)$. Take $F = \mathbb{F}_3$ with 'colors' $0, 1, 2 \in \mathbb{F}_3$ and suppose vertex 1 is colored by 0. In this case $f(x) = x(x-1)(x-2) = x^3 - x \in \mathbb{F}_3[x]$ and $g(x_i, x_j) = x_i^2 + x_i x_j + x_j^2 - 1$. If I is the ideal generated by $x_1, x_i^3 - x_i, 2 \leq i \leq 8$ and $g(x_i, x_j)$ for the edges (i, j) in $\mathcal{G}$, show that the reduced Gröbner basis for I with respect to the lexicographic monomial ordering $x_1 > x_2 > \cdots > x_8$ is $\{x_1, x_2, x_3 + x_8, x_4 + 2x_8, x_5 + x_8, x_6, x_7 + x_8, x_8^2 + 2\}$. Deduce that $\mathcal{G}$ has two distinct 3-colorings, determined by the coloring of vertex 8 (which must be colored by a nonzero element in $\mathbb{F}_3$), and exhibit the colorings of $\mathcal{G}$.

Show that if the edge $(3, 7)$ is added to $\mathcal{G}$ then the graph cannot be 3-colored.

(b) Take $F = \mathbb{F}_5$ with four 'colors' $1, 2, 3, 4 \in \mathbb{F}_5$, so $f(x) = x^4 - 1$ and we may use $g(x_i, x_j) = x_i^3 + x_i^2 x_j + x_i x_j^2 + x_j^3$. Show that the graph $\mathcal{G}$ with five vertices having 9 edges $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, $(4, 5)$ (the "complete graph on five vertices" with one edge removed) can be 4-colored but cannot be 3-colored.

(c) Use Gröbner bases to show that the graph $\mathcal{G}$ with nine vertices and 22 edges $(1, 4)$, $(1, 6)$, $(1, 7)$, $(1, 8)$, $(2, 3)$, $(2, 4)$, $(2, 6)$, $(2, 7)$, $(3, 5)$, $(3, 7)$, $(3, 9)$, $(4, 5)$, $(4, 6)$, $(4, 7)$, $(4, 9)$, $(5, 6)$, $(5, 7)$, $(5, 8)$, $(5, 9)$, $(6, 7)$, $(6, 9)$, $(7, 8)$ has precisely four 4-colorings up to a permutation of the colors (so a total of 96 total 4-colorings). Show that if the edge $(1, 5)$ is added then $\mathcal{G}$ cannot be 4-colored.