Math 502. Midterm assignment. Due Thursday March 3.

(1) Let G be a finite group. Prove that there is an isomorphism of $G \times G$-representations

$$L^2(G) \simeq \bigoplus_{(\pi,V)} V^* \otimes V,$$

where $L^2(G)$ is the space of all complex-valued functions on G, on which $G \times G$ acts by $(g_1, g_2) \cdot f(x) = f(g_1^{-1} x g_2)$. On the right-hand side, the sum is over a complete set of representatives of the isomorphism classes of irreducible representations of G.

Hint: use matrix coefficients.

The rest of this assignment is devoted to the background on modules that we will need.

Definition 1. Let R be a ring (not necessarily commutative, and not necessarily with an identity). A *left module* over R is an abelian group M endowed with action by R (denoted by $(r, m) \mapsto rm$), satisfying the following conditions (the group operation on M is denoted by $+$):

(1) $(r + s)m = rm + sm$ for all $r, s \in R$, $m \in M$;
(2) $(rs)m = r(sm)$ for all $r, s \in R$, $m \in M$;
(3) $r(m + n) = rm + rn$ for all $r \in R$, $m, n \in M$;
(4) if R has the identity element 1, then $1m = m$ for all $m \in M$.

The definition of a right R-module is similar.

Please read about modules in any abstract algebra text, e.g. Dummit and Foote "Abstract Algebra", 3d edition, 10.1–10.4. You'll need to feel comfortable with the following statements:

- $\mathbb{Z}$-modules are the same as abelian groups (and the homomorphisms of $\mathbb{Z}$-modules are homomorphisms of abelian groups);
- if $R = F$ is a field, then R-modules are the same as vector spaces over F (and the F-module homomorphisms are the same as the linear maps of vector spaces);
- if R is a polynomial ring $R = F[x]$ with F a field, then R-modules are the same as the pairs (V, A), where V is a vector space over F, and $A : V \to V$ is a linear operator (and the $F[x]$-module homomorphisms are the same as the maps $T : V \to V'$ such that $T \circ A = A' \circ T$).

We will also need the notion of an R-algebra:

Definition 2. Let R be a *commutative* ring with 1. Then an R-algebra is a ring A with an identity, equipped with a ring homomorphism $f : R \to A$ such that $f(1_R) = 1_A$, and the image $f(R)$ is contained in the centre of A (that is, $f(r)a = af(r)$ for every $a \in A$, $r \in R$).

Note that an R-algebra can be thought of as both left and right R-module: the module structure is given by $ra = ar = f(r)a$. This is the structure we refer to when we say that A is an R-module.

Please write up the following three exercises:

(2) Compute $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, where m, n are integers.
(3) Let A be a finite abelian group, think of it as a $\mathbb{Z}$-module. Prove that the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of A is zero: $\mathbb{Q} \otimes_{\mathbb{Z}} A = \{0\}$.
(4) Let R, S be commutative rings with $R \subset S$ and $1_S = 1_R$. Let I be an ideal in the polynomial ring $R[x_1, \ldots x_n]$. Prove that

$$S \otimes_R (R[x_1, \ldots x_n]/I) \simeq S[x_1, \ldots, x_n]/IS[x_1, \ldots, x_n]$$

as S-algebras.