Review topics for the final exam for Math 226

This is a rough list of the things you should definitely be able to do. I think the best way to use this review sheet is to try to recall or come up with an example for each concept or technique listed below. Then try to solve that example, and if you cannot (or could not come up with one to begin with), then re-read the corresponding section in the book. It is very important to try to think of all these concepts without opening the book, at first. Copies of any parts of this review sheet will not be allowed at the exam. Good luck!

1. Vectors, and equations of lines and planes

- Two ways to write a vector: $\langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- Basic operations with vectors: addition, subtraction, scalar multiplication.
- Finding a unit vector parallel to a given vector $\mathbf{a}$: $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$.
- Dot product and cross product of vectors.
- a component of $\mathbf{a}$ along $\mathbf{b}$. Vector projection of $\mathbf{a}$ onto $\mathbf{b}$.
- parametric and symmetric forms of the equation of a line in space (how to write an equation of the line that contains two given points). How to determine whether two lines in space intersect.
- Equations of planes: normal vector; how to write an equation of the plane through three given points.
- Finding distances: between two points, from a point to a plane, from a point to a line.
- How to find symmetric and parametric equations for the line of intersection of two planes.

2. Quadric surfaces and cylinders

This section is not explicitly on the exam. However, you need to be able to sketch things like cones, spheres, and circular paraboloids, since they are often used in problems about integration. Also, remember, that a "cylinder" means any surface that consists of lines parallel to a given line and passing through points on a given plane curve (so a cylinder doesn't have to be a circular cylinder).

3. Limits, Continuity, differentiability, partial derivatives, linear approximations

- Partial derivatives: the definition of partial derivatives of a function at a point. Remember problems similar to Problem 3 from Midterm 2, and problem 5 from take-home.
- Equality of mixed second-order partial derivatives (this holds under the condition that the second-order partials should exist and *are continuous* at the given point.
- The ϵ-δ definition of a limit, and of continuity of a function at a point. You need to be able to do examples with ϵ-δ similar to homework problems.
- The notions of continuity and differentiability for functions of two (or more) variables. You need to know the *definitions* of a continuous and especially of *differentiable* function of two variables well. You will also need to know (and understand) examples: of a function that is continuous in every direction but not continuous, and also of a function whose partial derivatives at a

point exist, but that is not differentiable. You might be asked to do proofs that rely on properties of continuous/differentiable functions.

- Linearizations: Remember that the linearization of the function $f(x, y)$ at the point (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$, and its graph $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is the tangent plane to the graph of the function $f(x, y)$ at the point (a, b).
- Using the linearization (or "differentials") to find the approximate value of the function, or the approximate error, etc.
- Chain rule (for functions of any number of variables); also, Chain rule using Jacobian matrices (as in Midterm 2 and written homework 4).
- Implicit differentiation and Implicit Function theorem. You need to be able to do implicit differentiation in all kinds of situations:
  (1) The easiest case: if z is an implicit function of x and y defined via some relation $F(x, y, z) = 0$. (if the right-hand side is not zero, take it all to the left). You need to find $\partial z/\partial x$ and $\partial z/\partial y$. Recall the formulas:
  $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}; \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z},$$
  You need to know how to prove them and and know how to use them. (To prove them, simply differentiate both sides of the relation defining $f(x, y)$ with respect to x, and every time you see z, treat it as a *function* of x and y (so use chain rule, product rule, etc.). Then you'll get a relation involving the unknown $\partial z/\partial x$. Then you can solve for it. (Also note that z can appear in the expressions you get for $\partial z/\partial x$ and $\partial z/\partial y$. To evaluate them at a given point, usually you need all three coordinates x, y, z).
  In this case the Implicit Function theorem says that z can be thought of as a differentiable implicit function of x, y provided $F_z \neq 0$.
  (2) You also need to be able to do implicit differentiation in the cases when there is a system of m equations in n variables. Then you get to choose m 'dependent' variables, and as long as the Jacobian determinant of partial derivatives with respect to these variables is not zero at a particular point, the Implicit Function theorem guarantees that your chosen variables are indeed differentiable implicit functions of the remaining variables in the neighbourhood of your point. This is discussed in a lot of detail in Homework 4, problems 5-7 (problem 5 has $n = 3$, $m = 2$, Problems 6-7 have $n = 4, m = 2$). Note that the "easy case" above is the case $m = 1$.
  (3) You need to be able to combine implicit differentiation with other techniques: linearization, chain rule, higher-order derivatives (as in Problems 6-7 (b) in Homework 4), or as in the problem about the second derivative in Midterm 2.
- Directional derivatives; the gradient vector. The main formula (for differentiable f):
  $$D_{\bar{u}} f = \nabla f \cdot \bar{u}.$$

Remember that this formula is based on chain rule and applies *only when f is differentiable*. (This could be an easy way to prove that a function is *not* differentiable, sometimes: if this formula fails, f cannot be differentiable at the given point). If f is not differentiable at a point (a, b) (or if you do not know whether it is differentiable), then the only way to find the directional derivative (if it exists) at this point is by using the definition

$$D_{\bar{u}}f|_{(a,b)} = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}.$$

Remember that $\bar{u}$ in all these formulas has to be a unit vector. If the vector defining the direction is not unit, you have to first make it unit. You need to understand the meaning (in words) of the directional derivative – the rate of change of f in the direction of $\bar{u}$.

- Because of the above formula, the direction of the fastest increase of a differentiable function (at a given point) is given by the gradient, and the direction of the fastest decrease is opposite to the gradient.
- You need to know how to use the gradient vector of a function of 3 variables to write an equation of the tangent plane to a surface given by some equation in x, y, z-variables.

  Note that we have two ways of writing the equation of a tangent plane to a graph of $f(x, y)$. One is using linearization of f (see above). The other (which is usually easier), is to use a gradient of a function of 3 variables. You can write $z = f(x, y)$, and make a new function $F(x, y, z) = f(x, y) - z$. Then the graph is the level surface given by $F(x, y, z) = 0$. Then you can use the gradient of F to find the normal vector to the tangent plane at a point $(a, b, f(a, b))$. A good exercise: prove to yourself that both methods of writing the equation of the tangent plane give the same plane (the equation looks a little different).
- You need to understand what the phrase "direction of the steepest ascent" means. Note: if the hill is the graph of a function $z = f(x, y)$, then the gradient of f is a 2-dimensional vector, so it gives a "compass direction" (which is the projection onto the xy-plane of the vector in 3 dimensions that gives your velocity when you are moving on that hill).

4. Critical points; Extremal problems

- Critical points and the second derivative test for functions of 2 variables: you need to know how to classify critical points of a function.
- You need to know what the Hessian matrix is and how it is used to give the second-order term in the Taylor approximation (the beginning of 12.9).
- Max/min of a function on a closed bounded domain (need to know how to check if the domain is closed – no proof required on the exam, just need to know that if the domain is defined by an *non-strict* inequality $f(x, y) \le c$ where f is continuous, then it is closed). The method is: find critical points inside and find max/min on the boundary, by any method (Lagrange or parameterising the boundary).
- Lagrange multipliers; including with more than one constraint.

5. Integration

- Iterated integrals. How to write an integral of a function of two variables over a domain in the plane, and how to change the order of integration.
- Double integrals in polar coordinates.

- **Jacobian change of variables formula for double integrals (as in Webwork 10).** You also need to be able to tell if the given transformation is one-to-one.
- Mean Value Theorem for integrals. The notion of "average" of a function.
- Improper integrals in two variables. You need to know the condition under which you can compute the double integral (or conclude that it diverges) by writing it as an iterated integral: (a) if the function is non-negative; or (b) if the iterated integral of its absolute value converges.
- Triple integrals: you need to be able to evaluate triple integrals if they are already given as an iterated integral; and need to know how to set up a triple integral as an iterated integral (that is, define the limits of integration correctly) for not very complicated solids, such as boxes (the fancier name for them is a rectangular prism), pyramids (the official name is tetrahedron), and similar things (to the extent of Webwork). You need to know how to change the order of integration in a triple integral.
- Triple integrals in cylindrical coordinates. This includes integration over solids such as circular paraboloids and cones, etc.
- Triple integrals in spherical coordinates. You need to be able to set up such integrals correctly over pieces of spheres and cones (as in webwork), and to switch between spherical and Cartesian coordinates.
- It is helpful to know how to write equations of:
  - In polar coordiantes on the plane:
    - a circle with centre at the origin
    - a circle with centre on the x-axis, or on the y-axis, and passing through the origin
    - a line through the origin
  - In cylindrical coordinates in space:
    - a cone
    - a circular paraboloid
    - a vertical plane passing through the origin
    - a sphere centred at the origin
  - In spherical coordinates:
    - cones around the z-axis
    - spheres centred at the origin
    - spheres with centre on the z-axis and passing through the origin
    - planes containing the z-axis

  Remember that when in doubt, you can take the equation in Caretsian coordinates, and simply use the conversion formulas to convert it to cylindrical or spherical.