Subgroups of the multiplicative group

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joint work with Lee Troupe

Analytic Number Theory
2017 CMS Winter Meeting
University of Waterloo
December 10, 2017

slides can be found on my web page
www.math.ubc.ca/~gerg/index.shtml?slides
Outline

1. Background, and distribution of \((\phi\text{-})\)additive functions
2. Results on the distribution of the number of subgroups of \(\mathbb{Z}_n^\times\)
3. Outline of the proofs
Our objects of study

**Definition**

The “multiplicative group” (or unit group) modulo $n$ is $\mathbb{Z}_n^\times = (\mathbb{Z}/n\mathbb{Z})^\times$, the group of reduced residue classes under multiplication (mod $n$).

$\mathbb{Z}_n^\times$ is some finite abelian group with $\phi(n)$ elements (usually not cyclic). Questions about its structure often turn into number theory (example: its exponent is the Carmichael $\lambda$-function).

**Overarching question (I heard it from Shparlinski)**

How many subgroups does $\mathbb{Z}_n^\times$ usually have?

**Notation (used throughout the talk)**

$I(n)$ is the number of isomorphism classes of subgroups of $\mathbb{Z}_n^\times$.

$G(n)$ is the number of subsets of $\mathbb{Z}_n^\times$ that are subgroups (that is, subgroups not up to isomorphism).
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The distribution of the number of subgroups of the multiplicative group
Distribution results: different strengths

By way of analogy: some historical results about the distribution of \( \omega(n) \), the number of distinct prime factors of \( n \).

- The average value of \( \omega(n) \) is \( \log \log n \).
  — requires an asymptotic formula for \( \sum_{n \leq x} \omega(n) \)
- The normal order (typical size) of \( \omega(n) \) is \( \log \log n \).
  — requires estimate for variance \( \sum_{n \leq x} \left( \omega(n) - \log \log n \right)^2 \)
- Erdős–Kac theorem: \( \omega(n) \) is asymptotically distributed like a normal random variable with mean \( \log \log n \) and variance \( \log \log n \). (More precise statement on next slide.)
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Erdős–Kac laws

**Definition**

A function $f(n)$ satisfies an Erdős–Kac law with mean $\mu(n)$ and variance $\sigma^2(n)$ if

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \leq x : \frac{f(n) - \mu(n)}{\sigma(n)} < u \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} \, dt$$

for every real number $u$.

**Standard notation**

$\omega(n)$ is the number of distinct prime factors of $n$.

$\Omega(n)$ is the number of prime factors of $n$ counted with multiplicity.

**Theorem (Erdős–Kac, 1940)**

Both $\omega(n)$ and $\Omega(n)$ satisfy Erdős–Kac laws with mean $\log \log n$ and variance $\log \log n$. 

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Other functions with Erdős–Kac laws

The paper of Erdős–Kac establishes these normal-distribution laws for a large class of additive functions: if \( n = p_1^{r_1} \cdots p_k^{r_k} \), then \( f(n) = f(p_1^{r_1}) + \cdots + f(p_k^{r_k}) \). Examples of non-additive functions:

Liu (2007)
On GRH, \( \omega(\#E(F_p)) \) satisfies an Erdős–Kac law with mean \( \frac{1}{2} \log \log p \) and variance \( \log \log p \).

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\( \omega(\phi(n)) \) and \( \Omega(\phi(n)) \) satisfy Erdős–Kac laws with mean \( \frac{1}{2}(\log \log n)^2 \) and variance \( \frac{1}{3}(\log \log n)^3 \).

\( \Omega(\phi(n)) \) is not additive, but is \( \phi \)-additive: if \( \phi(n) = p_1^{r_1} \cdots p_k^{r_k} \), then \( \Omega(\phi(n)) = \Omega(p_1^{r_1}) + \cdots + \Omega(p_k^{r_k}) \).
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The number of subgroups has a similar property

Reminder of notation

$I(n)$ is the number of isomorphism classes of subgroups of $\mathbb{Z}_n^\times$. $G(n)$ is the number of subsets of $\mathbb{Z}_n^\times$ that are subgroups.

Every finite abelian group is the direct sum of its $p$-Sylow subgroups, so consequently:

If $G_p(n)$ denotes the number of subgroups of the $p$-Sylow subgroup of $\mathbb{Z}_n^\times$, then

$G(n) = \prod_{p|\#\mathbb{Z}_n^\times} G_p(n) = \prod_{p|\phi(n)} G_p(n).$

And similarly for $I(n)$.

In particular, both $I(n)$ and $G(n)$ are “$\phi$-multiplicative” functions; so we might hope to get strong distributional information for the $\phi$-additive functions $\log I(n)$ and $\log G(n)$.
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Erdős–Kac laws for the number of subgroups

Theorem (M.-Troupe, submitted)

\[ \log I(n) \text{ satisfies an Erdős–Kac law with mean } \frac{\log 2}{2} (\log \log n)^2 \]
\[ \text{and variance } \frac{\log 2}{3} (\log \log n)^3. \]

How did we prove this?

We showed that \( \omega(\phi(n)) \log 2 \leq \log I(n) \leq \Omega(\phi(n)) \log 2 \), and then quoted Erdős–Pomerance.

Theorem (M.-Troupe, submitted)

\[ \log G(n) \text{ satisfies an Erdős–Kac law with mean } A (\log \log n)^2 \text{ and variance } C (\log \log n)^3, \text{ for certain constants } A \text{ and } C. \]

\[ \frac{\log 2}{2} \approx 0.34657 \text{ while } A \approx 0.72109, \text{ so typically } G(n) \approx I(n)^{2.08}. \]
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We had to look at these constants, so you do too

**Definition**

\[
A_0 = \frac{1}{4} \sum_p \frac{p^2 \log p}{(p - 1)^3(p + 1)}
\]

\[
A = \frac{\log 2}{2} + A_0 \approx 0.72109
\]

\[
B = \frac{1}{4} \sum_p \frac{p^3(p^4 - p^3 - p^3 - p - 1)(\log p)^2}{(p - 1)^6(p + 1)^2(p^2 + p + 1)}
\]

\[
C = \frac{(\log 2)^2}{3} + 2A_0 \log 2 + 4A_0^2 + B \approx 3.924
\]

(The two sums are convergent sums over all primes \( p \).)
How many subgroups can there be?

**Theorem (M.-Troupe, submitted)**

The order of magnitude of the maximal order of $\log I(n)$ is $\log n / \log \log n$. More precisely,

$$\frac{\log 2}{5} \frac{\log x}{\log \log x} \lesssim \max_{n \leq x} (\log I(n)) \lesssim \sqrt{2} \frac{\log x}{3 \log \log x}.$$

**Theorem (M.-Troupe, submitted)**

The order of magnitude of the maximal order of $\log G(n)$ is $(\log n)^2 / \log \log n$. More precisely,

$$\frac{1}{16} \frac{(\log x)^2}{\log \log x} \lesssim \max_{n \leq x} (\log G(n)) \lesssim \frac{1}{4} \frac{(\log x)^2}{\log \log x}.$$

**Consequence:** $G(n)$ can be superpolynomially large.

There are infinitely many integers $n$ with $G(n) > n^{2017}$!
How many subgroups can there be?

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Finite abelian groups and partitions

Facts about finite abelian $p$-groups

- Every finite abelian group of size $p^m$ can be written uniquely as $\mathbb{Z}_{p^\alpha} = \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_\ell}}$ for some partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of $m$ (so $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\ell$).
- So the number of isomorphism classes of subgroups of $\mathbb{Z}_{p^\alpha}$ is exactly the number of subpartitions $\beta \preceq \alpha$.

... which is somewhere between 2 and $2^m$ inclusive.

In other words:

\[
\log \#\{\text{subpartitions of } \alpha\} \text{ is between } \log 2 \text{ and } m \log 2.
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Finite abelian groups and partitions

Facts about finite abelian $p$-groups

- Every finite abelian group of size $p^m$ can be written uniquely as $\mathbb{Z}_{p^\alpha} = \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_\ell}}$ for some partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of $m$ (so $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\ell$).
- So the number of isomorphism classes of subgroups of $\mathbb{Z}_{p^\alpha}$ is exactly the number of subpartitions $\beta \preceq \alpha$ ...

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Application to distribution of $I(n)$

$I(n)$ is the number of isomorphism classes of subgroups of $\mathbb{Z}_n^\times$.

More notation

Let $\phi(n) = \prod_{p \mid \phi(n)} p^{m(p)}$, so that $\mathbb{Z}_n^\times \cong \bigoplus_{p \mid \phi(n)} \mathbb{Z}_{p^{\alpha(p)}}$ for some partitions $\alpha(p)$ of $m(p)$.

Then $\log I(n) = \sum_{p \mid \phi(n)} \log \# \{\text{subpartitions of } \alpha_p\}$ and hence

$$\sum_{p \mid \phi(n)} \log 2 \leq \log I(n) \leq \sum_{p \mid \phi(n)} m(p) \log 2$$

$$\omega(\phi(n)) \log 2 \leq \log I(n) \leq \Omega(\phi(n)) \log 2$$

Upper bound seems very wasteful, yet still good enough!

“Anatomy of integers” techniques show: most primes dividing $\phi(n)$ do so only once.
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"The distribution of the number of subgroups of the multiplicative group of $\mathbb{Z}_n$" by Greg Martin
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How many subgroups of each shape?

Notation: $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, $\mathbb{Z}_p^\alpha = \mathbb{Z}_p^{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_p^{\alpha_\ell}$

Definition

Given a subpartition $\beta$ of $\alpha$ and a prime $p$, define $N_p(\alpha, \beta)$ to be the number of subgroups inside $\mathbb{Z}_p^\alpha$ that are isomorphic to $\mathbb{Z}_p^\beta$.

Some classical exact formula (don’t read it)

Let $a = (a_1, a_2, \ldots, a_{\alpha_1})$ and $b = (b_1, b_2, \ldots, b_{\beta_1})$ be the conjugate partitions to $\alpha$ and $\beta$, respectively. Then

$$N_p(\alpha, \beta) = \prod_{j=1}^{\alpha_1} p^{(a_j - b_j)b_{j+1}} \left[ a_j - b_{j+1} \right]_{b_j - b_{j+1}}^p,$$

where $\left[ k \atop \ell \right]_p = \prod_{j=1}^{\ell} \frac{p^{k-\ell+j-1}}{p^j - 1}$ is the Gaussian binomial coefficient.
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The difference between algebra and analysis

\[ N_p(\alpha, \beta) = \prod_{j=1}^{\alpha_1} p^{(a_j - b_j)b_j + 1} \left[ \frac{a_j - b_j + 1}{b_j - b_{j+1}} \right]_p \]

is the number of subgroups inside \( \mathbb{Z}_{p^\alpha} \) isomorphic to \( \mathbb{Z}_{p^\beta} \).

It turns out that each factor is about \( p^{(a_j - b_j)b_j} \), which is maximally \( p^{a_j^2/4} \) when \( b_j = a_j/2 \), and is way smaller for noncentral values of \( b_j \). So the total number of subgroups inside \( \mathbb{Z}_{p^\alpha} \) is dominated by this special \( \beta = \frac{1}{2} \alpha \).

**Lemma**

*For any prime \( p \) and any partition \( \alpha \),*

\[ \log \# \{ \text{subgroups of } \mathbb{Z}_{p^\alpha} \} = \frac{\log p}{4} \sum_{j=1}^{\alpha_1} a_j^2 + O(\alpha_1 \log p). \]
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**Lemma**

For any prime \( p \) and any partition \( \alpha \),

\[
\log \# \{ \text{subgroups of } \mathbb{Z}_{p^\alpha} \} = \log p + \frac{1}{4} \sum_{j=1}^{\alpha_1} a_j^2 + O(\alpha_1 \log p).
\]
If \( \mathbb{Z}_n^\times \cong \bigoplus_{p|\phi(n)} \mathbb{Z}_{p^{\alpha(p)}} \), then which partition is \( \alpha(p) \)?

**Notation**

Let \( \omega_q(n) \) denote the number of distinct prime factors of \( n \) that are congruent to 1 (mod \( q \)).

**Answer (exact for odd squarefree \( n \), up to \( O(1) \) in general)**

\( \alpha(p) \) is the conjugate partition to \( (\omega_p(n), \omega_p^2(n), \ldots) \).

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\[
\log G_p(n) \approx \frac{\log p}{4} \sum_{j=1}^{\infty} \omega_{pj}(n)^2 \quad \text{for any prime } p \text{ dividing } \phi(n).
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Moreover, if \( p | \phi(n) \) and \( p^2 \nmid \phi(n) \), then \( \log G_p(n) = \log 2 \).

The distribution of the number of subgroups of the multiplicative group

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Sum the previous lemma over all primes

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For most integers \(n\), it’s acceptable to extend both sums over all primes dividing \(\phi(n)\) (the last sum should be suitably truncated):

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\log G(n) \approx \log 2 \cdot \omega(\phi(n)) + \frac{1}{4} \sum_{p^r} \omega_{p^r}(n)^2 \log p.
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Each function here has a known normal order; plugging in gives

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Final sketch

Getting beyond the normal order to an Erdős–Kac law requires computing all of the central moments of this approximation to $\log G(n)$. The correlations among the additive functions $\omega_q(n)$, and their correlations with $\omega(\phi(n))$, become important.

“Sieving and the Erdős–Kac theorem” (2007)

To compute the moments, we rely on a technique of Granville and Soundararajan to reduce the complexity of identifying the main terms of these moments.

Generalizing our method

Part of $\log G(n)$ is well approximated by a sum of squares of additive functions. Troupe and I (work in progress) can obtain an Erdős–Kac law for any fixed (nonnegative) polynomial evaluated at values of (appropriate) additive functions—for example, Erdős–Kac laws for products of additive functions.
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Our submitted paper “The distribution of the number of subgroups of the multiplicative group” and these slides are available for downloading.

**The paper with Lee Troupe**


**These slides**