Dense Egyptian fractions

Greg Martin
University of British Columbia

AMS Spring Central Sectional Meeting
University of Illinois at Urbana-Champaign
March 27, 2009
Outline

1. Introduction
2. Main theorem and proof
3. Surprise bonus
Egyptian fractions

**Definition**

Let $r$ be a positive rational number. An Egyptian fraction for $r$ is a sum of reciprocals of distinct positive integers that equals $r$.

**Example**

$$1 = 1/2 + 1/3 + 1/6$$

**Theorem (Fibonacci 1202, Sylvester 1880, …)**

Every positive rational number has an Egyptian fraction representation. (Proof: greedy algorithm.)

**Note:** we’ll restrict to $r = 1$ for most of the remainder of the talk; but everything holds true for any positive rational number $r$. 

Dense Egyptian fractions

Greg Martin
**Egyptian fractions**

**Definition**
Let $r$ be a positive rational number. An Egyptian fraction for $r$ is a sum of reciprocals of distinct positive integers that equals $r$.

**Example**

\[ 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \]

**Theorem (Fibonacci 1202, Sylvester 1880, ...)**
Every positive rational number has an Egyptian fraction representation. (Proof: greedy algorithm.)

**Note:** we’ll restrict to $r = 1$ for most of the remainder of the talk; but everything holds true for any positive rational number $r$. 

Dense Egyptian fractions

Greg Martin
Egyptian fractions

**Definition**
Let $r$ be a positive rational number. An Egyptian fraction for $r$ is a sum of reciprocals of distinct positive integers that equals $r$.

**Example**

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

**Theorem (Fibonacci 1202, Sylvester 1880, ...)**

*Every positive rational number has an Egyptian fraction representation.* (Proof: greedy algorithm.)

**Note:** we’ll restrict to $r = 1$ for most of the remainder of the talk; but everything holds true for any positive rational number $r$. 
Demoralizing Egyptian scribes

**Question**

How many terms can an Egyptian fraction for 1 have?

**Cheap answer**

Arbitrarily many, by the splitting trick:

\[
1 = 1/2 + 1/3 + 1/6 \\
= 1/2 + 1/3 + 1/7 + 1/(6 \times 7) \\
= 1/2 + 1/3 + 1/7 + 1/43 + 1/(42 \times 43) = \ldots
\]

But the denominators become enormous.

**Better question**

How many terms can an Egyptian fraction for 1 have, if the denominators are bounded by \(x\)?
Demoralizing Egyptian scribes

Question
How many terms can an Egyptian fraction for 1 have?

Cheap answer
Arbitrarily many, by the splitting trick:

\[
1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \\
= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{6 \times 7} \\
= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{42 \times 43} = \ldots
\]

But the denominators become enormous.

Better question
How many terms can an Egyptian fraction for 1 have, if the denominators are bounded by \(x\)?
Question

How many terms can an Egyptian fraction for 1 have?

Cheap answer

Arbitrarily many, by the splitting trick:

\[
1 = 1/2 + 1/3 + 1/6
= 1/2 + 1/3 + 1/7 + 1/(6 \times 7)
= 1/2 + 1/3 + 1/7 + 1/43 + 1/(42 \times 43) = \ldots
\]

But the denominators become enormous.

Better question

How many terms can an Egyptian fraction for 1 have, if the denominators are bounded by \( x \)?
Introduction

Main theorem and proof

Surprise bonus

Demoralizing Egyptian scribes

Question
How many terms can an Egyptian fraction for 1 have?

Cheap answer
Arbitrarily many, by the splitting trick:
\[
1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}
\]
\[
= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{(6 \times 7)}
\]
\[
= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{(42 \times 43)} = \ldots
\]
But the denominators become enormous.

Better question
How many terms can an Egyptian fraction for 1 have, if the denominators are bounded by \(x\)?
A simple example

\[ 1 = \sum_{n \in S} \frac{1}{n}, \text{ where:} \]


\[ S \] has 454 elements, all bounded by 1000.
A simple example

\[ 1 = \sum_{n \in S} \frac{1}{n}, \text{ where:} \]


\( S \) has 454 elements, all bounded by 1000
Suppose that there are \( t \) denominators, all bounded by \( x \), in an Egyptian fraction for 1. Then

\[
1 = \sum_{j=1}^{t} \frac{1}{n_j} \geq \sum_{n=x-t+1}^{x} \frac{1}{n} \sim \log \frac{x}{x-t}.
\]

So \( e \gtrsim \frac{x}{x-t} \), giving an upper bound for the number of terms:

\[
t \lesssim \left(1 - \frac{1}{e}\right)x.
\]
What’s best possible?

Suppose that there are $t$ denominators, all bounded by $x$, in an Egyptian fraction for 1. Then

$$1 = \sum_{j=1}^{t} \frac{1}{n_j} \geq \sum_{n=x-t+1}^{x} \frac{1}{n} \sim \log \frac{x}{x-t}.$$ 

So $e \gtrsim \frac{x}{x-t}$, giving an upper bound for the number of terms:

$$t \lesssim \left(1 - \frac{1}{e}\right)x.$$
Suppose that there are $t$ denominators, all bounded by $x$, in an Egyptian fraction for 1. Then

$$1 = \sum_{j=1}^{t} \frac{1}{n_j} \geq \sum_{n=x-t+1}^{x} \frac{1}{n} \sim \log \frac{x}{x-t}.$$ 

So $e \geq \frac{x}{x-t}$, giving an upper bound for the number of terms:

$$t \lesssim \left(1 - \frac{1}{e}\right)x.$$
Suppose that there are $t$ denominators, all bounded by $x$, in an Egyptian fraction for 1. Then

$$1 = \sum_{j=1}^{t} \frac{1}{n_j} \geq \sum_{n=x-t+1}^{x} \frac{1}{n} \sim \log \frac{x}{x-t}.$$ 

So $e \gtrsim \frac{x}{x-t}$, giving an upper bound for the number of terms:

$$t \lesssim \left(1 - \frac{1}{e}\right)x.$$
Lemma ("No tiny multiples of huge primes")

If a prime $p$ divides a denominator in an Egyptian fraction for 1 whose denominators are at most $x$, then $p \lesssim x / \log x$.

Proof

- If $pd_1, \ldots, pd_j$ are all the denominators that are divisible by $p$, then $\frac{1}{pd_1} + \cdots + \frac{1}{pd_j}$ can't have $p$ dividing the denominator when reduced to lowest terms.
- Its numerator $\text{lcm}[d_1, \ldots, d_j](\frac{1}{d_1} + \cdots + \frac{1}{d_j})$ is a multiple of $p$.
- If $M = \max\{d_1, \ldots, d_j\}$, then
  $$p \lesssim \text{lcm}[1, \ldots, M] \log M < e^{(1+o(1))M}.$$ 
- Therefore $\log p \lesssim M \leq \frac{x}{p}$.
Even better best possible

Lemma (“No tiny multiples of huge primes”)

If a prime $p$ divides a denominator in an Egyptian fraction for $1$ whose denominators are at most $x$, then $p \lesssim \frac{x}{\log x}$.

Proof

1. If $pd_1, \ldots, pd_j$ are all the denominators that are divisible by $p$, then $\frac{1}{pd_1} + \cdots + \frac{1}{pd_j}$ can’t have $p$ dividing the denominator when reduced to lowest terms.
2. Its numerator $\text{lcm}[d_1, \ldots, d_j](\frac{1}{d_1} + \cdots + \frac{1}{d_j})$ is a multiple of $p$.
3. If $M = \max\{d_1, \ldots, d_j\}$, then
   \[ p \lesssim \text{lcm}[1, \ldots, M] \log M < e^{(1+o(1))M}. \]
4. Therefore $\log p \lesssim M \leq \frac{x}{p}$.  

Dense Egyptian fractions

Greg Martin
Even better best possible

**Lemma (“No tiny multiples of huge primes”)**

If a prime $p$ divides a denominator in an Egyptian fraction for $1$ whose denominators are at most $x$, then $p \lesssim x / \log x$.

**Proof**

- If $pd_1, \ldots, pd_j$ are all the denominators that are divisible by $p$, then $\frac{1}{pd_1} + \cdots + \frac{1}{pd_j}$ can’t have $p$ dividing the denominator when reduced to lowest terms.
- Its numerator $\text{lcm}[d_1, \ldots, d_j](\frac{1}{d_1} + \cdots + \frac{1}{d_j})$ is a multiple of $p$.
- If $M = \max\{d_1, \ldots, d_j\}$, then
  $$p \lesssim \text{lcm}[1, \ldots, M] \log M < e^{(1+o(1))M}.$$  
- Therefore $\log p \lesssim M \leq \frac{x}{p}$.  

Dense Egyptian fractions  
Greg Martin
Lemma ("No tiny multiples of huge primes")

If a prime $p$ divides a denominator in an Egyptian fraction for $1$ whose denominators are at most $x$, then $p \lesssim x/\log x$.

Proof

- If $pd_1, \ldots, pd_j$ are all the denominators that are divisible by $p$, then $\frac{1}{pd_1} + \cdots + \frac{1}{pd_j}$ can’t have $p$ dividing the denominator when reduced to lowest terms.
- Its numerator $\text{lcm}[d_1, \ldots, d_j](\frac{1}{d_1} + \cdots + \frac{1}{d_j})$ is a multiple of $p$.
- If $M = \max\{d_1, \ldots, d_j\}$, then
  $$p \lesssim \text{lcm}[1, \ldots, M] \log M < e^{(1+o(1))M}.$$ 
- Therefore $\log p \lesssim M \leq \frac{x}{p}$. 

Dense Egyptian fractions
Even better best possible

**Lemma (“No tiny multiples of huge primes”)**

*If a prime $p$ divides a denominator in an Egyptian fraction for 1 whose denominators are at most $x$, then $p \ll \frac{x}{\log x}$.***

**Proof**

- If $pd_1, \ldots, pd_j$ are all the denominators that are divisible by $p$, then $\frac{1}{pd_1} + \cdots + \frac{1}{pd_j}$ can’t have $p$ dividing the denominator when reduced to lowest terms.
- Its numerator $\text{lcm}[d_1, \ldots, d_j] \left( \frac{1}{d_1} + \cdots + \frac{1}{d_j} \right)$ is a multiple of $p$.
- If $M = \max\{d_1, \ldots, d_j\}$, then
  \[
  p \ll \text{lcm}[1, \ldots, M] \log M < e^{(1+o(1))M}.
  \]
- Therefore $\log p \ll M \leq \frac{x}{p}$. 
Even better best possible

Lemma ("No tiny multiples of huge primes")

If a prime $p$ divides a denominator in an Egyptian fraction for $1$ whose denominators are at most $x$, then $p \ll x / \log x$.

Proof

- If $pd_1, \ldots, pd_j$ are all the denominators that are divisible by $p$, then $\frac{1}{pd_1} + \cdots + \frac{1}{pd_j}$ can’t have $p$ dividing the denominator when reduced to lowest terms.
- Its numerator $\text{lcm}[d_1, \ldots, d_j](\frac{1}{d_1} + \cdots + \frac{1}{d_j})$ is a multiple of $p$.
- If $M = \max\{d_1, \ldots, d_j\}$, then $p \ll \text{lcm}[1, \ldots, M] \log M < e^{(1+o(1))M}$.
- Therefore $\log p \ll M \leq \frac{x}{p}$.
Even better best possible

Lemma ("No tiny multiples of huge primes")

If a prime $p$ divides a denominator in an Egyptian fraction for 1 whose denominators are at most $x$, then $p \lesssim x / \log x$.

Note: most places in this talk, when I say “prime” I really should be saying “prime power”.

Using this lemma, it’s easy to show that the number $t$ of terms in an Egyptian fraction for 1 whose denominators are at most $x$ satisfies

$$t \lesssim \left(1 - \frac{1}{e}\right)x - \delta \frac{x \log \log x}{\log x} \quad \text{for some} \ \delta > 0.$$
Even better best possible

Lemma ("No tiny multiples of huge primes")

*If a prime $p$ divides a denominator in an Egyptian fraction for 1 whose denominators are at most $x$, then $p \lesssim \frac{x}{\log x}$.*

**Note:** most places in this talk, when I say “prime” I really should be saying “prime power”.

Using this lemma, it’s easy to show that the number $t$ of terms in an Egyptian fraction for 1 whose denominators are at most $x$ satisfies

$$t \lesssim \left(1 - \frac{1}{e}\right)x - \delta \frac{x \log \log x}{\log x}$$

for some $\delta > 0$. 
Dense Egyptian fractions

**Theorem (M., 2000)**

Given $x \geq 6$, there is an Egyptian fraction for 1 with

$$(1 - \frac{1}{e})x + O(x \log \log x / \log x)$$

terms and every denominator bounded by $x$.

**Method of proof (Croot; M.)**

- Start with a large set $S$ of integers not exceeding $x$ so that $\frac{A}{B} = \sum_{n \in S} \frac{1}{n}$ is approximately 1.
- Considering the primes $q$ dividing $B$ one by one, delete or add a few terms of $S$ so that $q$ doesn’t divide the new denominator $B$. Make the deleted/added elements large, so that their small reciprocals don’t affect the sum much.
- Sincerely hope that everything works out in the end.
Dense Egyptian fractions

**Theorem (M., 2000)**

Given \( x \geq 6 \), there is an Egyptian fraction for 1 with
\[
(1 - \frac{1}{e})x + O(x \log \log x / \log x)
\]
terms and every denominator bounded by \( x \).

**Method of proof (Croot; M.)**

- Start with a large set \( S \) of integers not exceeding \( x \) so that
\[
\frac{A}{B} = \sum_{n \in S} \frac{1}{n}
\]
is approximately 1.
- Considering the primes \( q \) dividing \( B \) one by one, delete or add a few terms of \( S \) so that \( q \) doesn’t divide the new denominator \( B \). Make the deleted/added elements large, so that their small reciprocals don’t affect the sum much.
- Sincerely hope that everything works out in the end.
Dense Egyptian fractions

Theorem (M., 2000)

Given $x \geq 6$, there is an Egyptian fraction for 1 with $(1 - \frac{1}{e})x + O(x \log \log x / \log x)$ terms and every denominator bounded by $x$.

Method of proof (Croot; M.)

- Start with a large set $S$ of integers not exceeding $x$ so that $\frac{A}{B} = \sum_{n \in S} \frac{1}{n}$ is approximately 1.
- Considering the primes $q$ dividing $B$ one by one, delete or add a few terms of $S$ so that $q$ doesn’t divide the new denominator $B$. Make the deleted/added elements large, so that their small reciprocals don’t affect the sum much.
- Sincerely hope that everything works out in the end.
**Theorem (M., 2000)**

*Given* $x \geq 6$, *there is an Egyptian fraction for* $1$ *with*

$$(1 - \frac{1}{e})x + O(x \log \log x / \log x)$$

*terms and every denominator bounded by* $x$.

**Method of proof (Croot; M.)**

- Start with a large set $S$ of integers not exceeding $x$ so that

$$\frac{A}{B} = \sum_{n \in S} \frac{1}{n}$$

is approximately 1.

- Considering the primes $q$ dividing $B$ one by one, delete or add a few terms of $S$ so that $q$ doesn’t divide the new denominator $B$. Make the deleted/added elements large, so that their small reciprocals don’t affect the sum much.

- Sincerely hope that everything works out in the end.
## Dense Egyptian fractions

### Theorem (M., 2000)

Given \( x \geq 6 \), there is an Egyptian fraction for 1 with

\[
(1 - \frac{1}{e})x + O(x \log \log x / \log x)
\]

terms and every denominator bounded by \( x \).

### Method of proof (Croot; M.)

- Start with a large set \( S \) of integers not exceeding \( x \) so that
  \[
  \frac{A}{B} = \sum_{n \in S} \frac{1}{n}
  \]
  is approximately 1.

- Considering the primes \( q \) dividing \( B \) one by one, delete or add a few terms of \( S \) so that \( q \) doesn’t divide the new denominator \( B \). Make the deleted/added elements large, so that their small reciprocals don’t affect the sum much.

- Sincerely hope that everything works out in the end.
A desired congruence

Definition

Given an Egyptian fraction \( \frac{A}{B} = \sum_{n \in S} \frac{1}{n} \) and a prime \( q \) dividing \( B \), define \( a \equiv A \left( \frac{B}{q} \right)^{-1} \pmod{q} \).

- When deleting elements from \( S \): want to find a set \( K \) such that \( qK \subset S \) and \( \sum_{m \in K} m^{-1} \equiv a \pmod{q} \). Then the denominator of \( \sum_{n \in S \setminus qK} \frac{1}{n} = \frac{A}{B} - \sum_{m \in K} \frac{1}{qm} \) is no longer divisible by \( q \).
- When adding elements to \( S \): want to find a set \( K \) such that \( qK \cap S = \emptyset \) and \( \sum_{m \in K} m^{-1} \equiv -a \pmod{q} \).
- To keep all new terms distinct, make sure the prime factors of the elements of \( K \) are always less than \( q \).

Notation: \( qK = \{qm : m \in K\} \)
A desired congruence

Definition

Given an Egyptian fraction \( \frac{A}{B} = \sum_{n \in S} \frac{1}{n} \) and a prime \( q \) dividing \( B \), define \( a \equiv A \left( \frac{B}{q} \right)^{-1} \pmod{q} \).

- When deleting elements from \( S \): want to find a set \( K \) such that \( qK \subset S \) and \( \sum_{m \in K} m^{-1} \equiv a \pmod{q} \). Then the denominator of \( \sum_{n \in S \setminus qK} \frac{1}{n} = \frac{A}{B} - \sum_{m \in K} \frac{1}{qm} \) is no longer divisible by \( q \).

- When adding elements to \( S \): want to find a set \( K \) such that \( qK \cap S = \emptyset \) and \( \sum_{m \in K} m^{-1} \equiv -a \pmod{q} \).

- To keep all new terms distinct, make sure the prime factors of the elements of \( K \) are always less than \( q \).

Notation: \( qK = \{qm: m \in K\} \)
A desired congruence

**Definition**

Given an Egyptian fraction $\frac{A}{B} = \sum_{n \in S} \frac{1}{n}$ and a prime $q$ dividing $B$, define $a \equiv A(B/q)^{-1} \pmod{q}$.

- When deleting elements from $S$: want to find a set $\mathcal{K}$ such that $q\mathcal{K} \subset S$ and $\sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q}$. Then the denominator of $\sum_{n \in S \setminus q\mathcal{K}} \frac{1}{n} = \frac{A}{B} - \sum_{m \in \mathcal{K}} \frac{1}{qm}$ is no longer divisible by $q$.
- When adding elements to $S$: want to find a set $\mathcal{K}$ such that $q\mathcal{K} \cap S = \emptyset$ and $\sum_{m \in \mathcal{K}} m^{-1} \equiv -a \pmod{q}$.
- To keep all new terms distinct, make sure the prime factors of the elements of $\mathcal{K}$ are always less than $q$.

**Notation:** $q\mathcal{K} = \{qm : m \in \mathcal{K}\}$
A desired congruence

**Definition**

Given an Egyptian fraction \( \frac{A}{B} = \sum_{n \in S} \frac{1}{n} \) and a prime \( q \) dividing \( B \), define \( a \equiv A(B/q)^{-1} \pmod{q} \).

- When deleting elements from \( S \): want to find a set \( \mathcal{K} \) such that \( q\mathcal{K} \subset S \) and \( \sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q} \). Then the denominator of \( \sum_{n \in S \setminus q\mathcal{K}} \frac{1}{n} = \frac{A}{B} - \sum_{m \in \mathcal{K}} \frac{1}{qm} \) is no longer divisible by \( q \).
- When adding elements to \( S \): want to find a set \( \mathcal{K} \) such that \( q\mathcal{K} \cap S = \emptyset \) and \( \sum_{m \in \mathcal{K}} m^{-1} \equiv -a \pmod{q} \).
- To keep all new terms distinct, make sure the prime factors of the elements of \( \mathcal{K} \) are always less than \( q \).

**Notation:** \( q\mathcal{K} = \{qm : m \in \mathcal{K}\} \)
A desired congruence

**Definition**

Given an Egyptian fraction \( \frac{A}{B} = \sum_{n \in S} \frac{1}{n} \) and a prime \( q \) dividing \( B \), define \( a \equiv A(B/q)^{-1} \pmod{q} \).

- When deleting elements from \( S \): want to find a set \( K \) such that \( qK \subset S \) and \( \sum_{m \in K} m^{-1} \equiv a \pmod{q} \). Then the denominator of \( \sum_{n \in S \setminus qK} \frac{1}{n} = \frac{A}{B} - \sum_{m \in K} \frac{1}{qm} \) is no longer divisible by \( q \).
- When adding elements to \( S \): want to find a set \( K \) such that \( qK \cap S = \emptyset \) and \( \sum_{m \in K} m^{-1} \equiv -a \pmod{q} \).
- To keep all new terms distinct, make sure the prime factors of the elements of \( K \) are always less than \( q \).

**Notation:** \( qK = \{qm: m \in K\} \)
Adapting Croot’s technique

**Proposition (suitable for large primes $q$)**

*Given a prime $q$, let $\log q < B < q$. Let $\mathcal{M}$ be a set of at least $B^{2/3} (\log q)^2$ integers not exceeding $B$, each of which is of the form $p_1p_2$. Then for any integer $a$, there exists a subset $\mathcal{K}$ of $\mathcal{M}$ such that $\sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q}$.*

**Proof:** The number of such subsets equals (with $e(x) = e^{2\pi ix}$)

$$
\sum_{\mathcal{K} \subset \mathcal{M}} \frac{1}{q} \sum_{h=0}^{q-1} e\left(\frac{h}{q} \left( \sum_{m \in \mathcal{K}} m^{-1} - a \right) \right) = \frac{1}{q} \sum_{h=0}^{q-1} e\left(-\frac{ha}{q}\right) \prod_{m \in \mathcal{M}} \left(1 + e\left(\frac{hm^{-1}}{q}\right)\right).
$$

A pigeonhole argument (on the divisors of some auxiliary integer $A$, which is where the form $p_1p_2$ is used) shows that for $h \neq 0$, lots of the $hm^{-1} \pmod{q}$ must be reasonably far from 0, which gives cancellation in the product.
Proposition (suitable for large primes $q$)

Given a prime $q$, let $\log q < B < q$. Let $\mathcal{M}$ be a set of at least $B^{2/3}(\log q)^2$ integers not exceeding $B$, each of which is of the form $p_1 p_2$. Then for any integer $a$, there exists a subset $\mathcal{K}$ of $\mathcal{M}$ such that $\sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q}$.

Proof: The number of such subsets equals (with $e(x) = e^{2\pi i x}$)

$$
\sum_{\mathcal{K} \subseteq \mathcal{M}} \frac{1}{q} \sum_{h=0}^{q-1} e \left( \frac{h}{q} \left( \sum_{m \in \mathcal{K}} m^{-1} - a \right) \right) = \frac{1}{q} \sum_{h=0}^{q-1} e \left( -\frac{ha}{q} \right) \prod_{m \in \mathcal{M}} \left( 1 + e \left( \frac{hm^{-1}}{q} \right) \right).
$$

A pigeonhole argument (on the divisors of some auxiliary integer $A$, which is where the form $p_1 p_2$ is used) shows that for $h \neq 0$, lots of the $hm^{-1} \pmod{q}$ must be reasonably far from 0, which gives cancellation in the product.
Adapting Croot’s technique

Proposition (suitable for large primes $q$)

*Given a prime $q$, let $\log q < B < q$. Let $\mathcal{M}$ be a set of at least $B^{2/3}(\log q)^2$ integers not exceeding $B$, each of which is of the form $p_1p_2$. Then for any integer $a$, there exists a subset $\mathcal{K}$ of $\mathcal{M}$ such that $\sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q}$.***

**Proof:** The number of such subsets equals (with $e(x) = e^{2\pi i x}$)

$$
\sum_{\mathcal{K} \subset \mathcal{M}} \frac{1}{q} \sum_{h=0}^{q-1} e \left( \frac{h}{q} \left( \sum_{m \in \mathcal{K}} m^{-1} - a \right) \right) = \frac{1}{q} \sum_{h=0}^{q-1} e \left( -\frac{ha}{q} \right) \prod_{m \in \mathcal{M}} \left( 1 + e \left( \frac{hm^{-1}}{q} \right) \right).
$$

A pigeonhole argument (on the divisors of some auxiliary integer $A$, which is where the form $p_1p_2$ is used) shows that for $h \neq 0$, lots of the $hm^{-1} \pmod{q}$ must be reasonably far from 0, which gives cancellation in the product.
Adapting Croot’s technique

Proposition (suitable for large primes $q$)

Given a prime $q$, let $\log q < B < q$. Let $\mathcal{M}$ be a set of at least $B^{2/3} (\log q)^2$ integers not exceeding $B$, each of which is of the form $p_1 p_2$. Then for any integer $a$, there exists a subset $\mathcal{K}$ of $\mathcal{M}$ such that $\sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q}$.

Proof: The number of such subsets equals (with $e(x) = e^{2\pi i x}$)

$$\sum_{\mathcal{K} \subseteq \mathcal{M}} \frac{1}{q} \sum_{h=0}^{q-1} e \left( \frac{h}{q} \left( \sum_{m \in \mathcal{K}} m^{-1} - a \right) \right) = \frac{1}{q} \sum_{h=0}^{q-1} e \left( -\frac{ha}{q} \right) \prod_{m \in \mathcal{M}} \left( 1 + e \left( \frac{hm^{-1}}{q} \right) \right).$$

A pigeonhole argument (on the divisors of some auxiliary integer $A$, which is where the form $p_1 p_2$ is used) shows that for $h \neq 0$, lots of the $hm^{-1} \pmod{q}$ must be reasonably far from 0, which gives cancellation in the product.
Proposition (suitable for large primes $q$)

Given a prime $q$, let $\log q < B < q$. Let $\mathcal{M}$ be a set of at least $B^{2/3} (\log q)^2$ integers not exceeding $B$, each of which is of the form $p_1p_2$. Then for any integer $a$, there exists a subset $\mathcal{K}$ of $\mathcal{M}$ such that $\sum_{m \in \mathcal{K}} m^{-1} \equiv a \pmod{q}$.

Proof: The number of such subsets equals (with $e(x) = e^{2\pi ix}$)

$$
\sum_{\mathcal{K} \subset \mathcal{M}} \frac{1}{q} \sum_{h=0}^{q-1} e\left(\frac{h}{q} \left( \sum_{m \in \mathcal{K}} m^{-1} - a \right) \right) = \frac{1}{q} \sum_{h=0}^{q-1} e\left(-\frac{ha}{q}\right) \prod_{m \in \mathcal{M}} \left(1 + e\left(\frac{hm^{-1}}{q}\right)\right).
$$

A pigeonhole argument (on the divisors of some auxiliary integer $A$, which is where the form $p_1p_2$ is used) shows that for $h \neq 0$, lots of the $hm^{-1} \pmod{q}$ must be reasonably far from 0, which gives cancellation in the product.
Small prime powers, explicitly

For the small prime powers \( q_1 = 2, q_2 = 3, q_3 = 4, \ldots \), we add to \( S \) the single denominator \( n_j = \text{lcm}[1, \ldots, q_j]/b \), where \( b \in [1, q_j - 1] \) is chosen to make the earlier congruence hold.

**Denominators are small enough, but not too small**

- The \( n_j \) are less than \( x \) when \( q_j < (1 - \varepsilon) \log x \), say.
- Since \( n_j \) is at least \( \text{lcm}[1, \ldots, q_j]/(q_j - 1) \), the sum of their reciprocals is (as Croot observed) less than the telescoping sum

\[
\sum_{j=1}^{\infty} \frac{q_j - 1}{\text{lcm}[1, \ldots, q_j]} = \sum_{j=1}^{\infty} \left( \frac{1}{\text{lcm}[1, \ldots, q_{j-1}]} - \frac{1}{\text{lcm}[1, \ldots, q_j]} \right) = 1.
\]
Small prime powers, explicitly

For the small prime powers \( q_1 = 2, q_2 = 3, q_3 = 4, \ldots \), we add to \( S \) the single denominator \( n_j = \text{lcm}[1, \ldots, q_j]/b \), where \( b \in [1, q_j - 1] \) is chosen to make the earlier congruence hold.

**Denominators are small enough, but not too small**

- The \( n_j \) are less than \( x \) when \( q_j < (1 - \varepsilon) \log x \), say.
- Since \( n_j \) is at least \( \text{lcm}[1, \ldots, q_j]/(q_j - 1) \), the sum of their reciprocals is (as Croot observed) less than the telescoping sum

\[
\sum_{j=1}^{\infty} \frac{q_j - 1}{\text{lcm}[1, \ldots, q_j]} = \sum_{j=1}^{\infty} \left( \frac{1}{\text{lcm}[1, \ldots, q_{j-1}]} - \frac{1}{\text{lcm}[1, \ldots, q_j]} \right) = 1.
\]
Small prime powers, explicitly

For the small prime powers $q_1 = 2$, $q_2 = 3$, $q_3 = 4$, ..., we add to $S$ the single denominator $n_j = \frac{\text{lcm}[1, \ldots, q_j]}{b}$, where $b \in [1, q_j - 1]$ is chosen to make the earlier congruence hold.

**Denominators are small enough, but not too small**

- The $n_j$ are less than $x$ when $q_j < (1 - \varepsilon) \log x$, say.
- Since $n_j$ is at least $\frac{\text{lcm}[1, \ldots, q_j]}{(q_j - 1)}$, the sum of their reciprocals is (as Croot observed) less than the telescoping sum

$$
\sum_{j=1}^{\infty} \frac{q_j - 1}{\text{lcm}[1, \ldots, q_j]} = \sum_{j=1}^{\infty} \left( \frac{1}{\text{lcm}[1, \ldots, q_{j-1}]} - \frac{1}{\text{lcm}[1, \ldots, q_j]} \right) = 1.
$$
The construction

To start: Let $S$ be the set of all integers between $\frac{x}{e}$ and $x$ that are not divisible by a prime larger than $x/(\log x)^{22}$.

- Cardinality of $S$ is $(1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $\sum_{n \in S} \frac{1}{n} \sim 1 - 22 \log \log x / \log x$

Large $q$: Delete a few elements from $S$ for every large prime, by the earlier proposition.

- In all, delete $O(x/\log x)$ elements from the original $S$
- $\sum_{n \in S} \frac{1}{n} \lesssim 1 - 22 \log \log x / \log x$

Small $q$: Finally, add at most 1 element to $S$ for every small prime, as in the previous slide.

- Final cardinality of $S$ is $\gtrsim (1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $0 < \sum_{n \in S} \frac{1}{n} \lesssim (1 - 22 \log \log x / \log x) + 1 < 2$
- Denominator of $\sum_{n \in S} \frac{1}{n}$ is not divisible by any prime

Conclusion: $\sum_{n \in S} \frac{1}{n} = 1!$
The construction

To start: Let $S$ be the set of all integers between $\frac{x}{e}$ and $x$ that are not divisible by a prime larger than $x/(\log x)^{22}$.

- Cardinality of $S$ is $(1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $\sum_{n \in S} \frac{1}{n} \sim 1 - 22 \log \log x / \log x$

Large $q$: Delete a few elements from $S$ for every large prime, by the earlier proposition.

- In all, delete $O(x / \log x)$ elements from the original $S$
- $\sum_{n \in S} \frac{1}{n} \lesssim 1 - 22 \log \log x / \log x$

Small $q$: Finally, add at most 1 element to $S$ for every small prime, as in the previous slide.

- Final cardinality of $S$ is $\gtrsim (1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $0 < \sum_{n \in S} \frac{1}{n} \lesssim (1 - 22 \log \log x / \log x) + 1 < 2$
- Denominator of $\sum_{n \in S} \frac{1}{n}$ is not divisible by any prime

Conclusion: $\sum_{n \in S} \frac{1}{n} = 1!$
The construction

To start: Let $S$ be the set of all integers between $\frac{x}{e}$ and $x$ that are not divisible by a prime larger than $x/\left(\log x\right)^{22}$.

- Cardinality of $S$ is $(1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $\sum_{n \in S} \frac{1}{n} \sim 1 - 22 \log \log x / \log x$

Large $q$: Delete a few elements from $S$ for every large prime, by the earlier proposition.

- In all, delete $O(x/ \log x)$ elements from the original $S$
- $\sum_{n \in S} \frac{1}{n} \lesssim 1 - 22 \log \log x / \log x$

Small $q$: Finally, add at most 1 element to $S$ for every small prime, as in the previous slide.

- Final cardinality of $S$ is $\gtrsim (1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $0 \lesssim \sum_{n \in S} \frac{1}{n} \lesssim (1 - 22 \log \log x / \log x) + 1 < 2$
- Denominator of $\sum_{n \in S} \frac{1}{n}$ is not divisible by any prime

Conclusion: $\sum_{n \in S} \frac{1}{n} = 1!$
The construction

To start: Let $S$ be the set of all integers between $\frac{x}{e}$ and $x$ that are not divisible by a prime larger than $x/(\log x)^{22}$.

- Cardinality of $S$ is $(1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $\sum_{n \in S} \frac{1}{n} \sim 1 - 22 \log \log x / \log x$

Large $q$: Delete a few elements from $S$ for every large prime, by the earlier proposition.

- In all, delete $O(x/\log x)$ elements from the original $S$
- $\sum_{n \in S} \frac{1}{n} \lesssim 1 - 22 \log \log x / \log x$

Small $q$: Finally, add at most 1 element to $S$ for every small prime, as in the previous slide.

- Final cardinality of $S$ is $\gtrsim (1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $0 < \sum_{n \in S} \frac{1}{n} \lesssim (1 - 22 \log \log x / \log x) + 1 < 2$
- Denominator of $\sum_{n \in S} \frac{1}{n}$ is not divisible by any prime

Conclusion: $\sum_{n \in S} \frac{1}{n} = 1!$
The construction

To start: Let $S$ be the set of all integers between $\frac{x}{e}$ and $x$ that are not divisible by a prime larger than $x/(\log x)^{22}$.

- Cardinality of $S$ is $(1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $\sum_{n \in S} \frac{1}{n} \sim 1 - 22 \log \log x / \log x$

Large $q$: Delete a few elements from $S$ for every large prime, by the earlier proposition.

- In all, delete $O(x/ \log x)$ elements from the original $S$
- $\sum_{n \in S} \frac{1}{n} \lesssim 1 - 22 \log \log x / \log x$

Small $q$: Finally, add at most 1 element to $S$ for every small prime, as in the previous slide.

- Final cardinality of $S$ is $\gtrsim (1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $0 < \sum_{n \in S} \frac{1}{n} \lesssim (1 - 22 \log \log x / \log x) + 1 < 2$
- Denominator of $\sum_{n \in S} \frac{1}{n}$ is not divisible by any prime

Conclusion: $\sum_{n \in S} \frac{1}{n} = 1!$
The construction

To start: Let $S$ be the set of all integers between $\frac{x}{e}$ and $x$ that are not divisible by a prime larger than $x/(\log x)^{22}$.
- Cardinality of $S$ is $(1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $\sum_{n \in S} \frac{1}{n} \sim 1 - 22 \log \log x / \log x$

Large $q$: Delete a few elements from $S$ for every large prime, by the earlier proposition.
- In all, delete $O(x / \log x)$ elements from the original $S$
- $\sum_{n \in S} \frac{1}{n} \lesssim 1 - 22 \log \log x / \log x$

Small $q$: Finally, add at most 1 element to $S$ for every small prime, as in the previous slide.
- Final cardinality of $S$ is $\gtrsim (1 - \frac{1}{e})x + O(x \log \log x / \log x)$
- $0 < \sum_{n \in S} \frac{1}{n} \lesssim (1 - 22 \log \log x / \log x) + 1 < 2$
- Denominator of $\sum_{n \in S} \frac{1}{n}$ is not divisible by any prime

Conclusion: $\sum_{n \in S} \frac{1}{n} = 1!$
The construction

To start: Let $S$ be the set of all integers between $\frac{x}{e}$ and $x$ that are not divisible by a prime larger than $x/(\log x)^{22}$.

- Cardinality of $S$ is $(1 - \frac{1}{e})x + O(x \log \log x/\log x)$
- $\sum_{n \in S} \frac{1}{n} \sim 1 - 22 \log \log x/\log x$

Large $q$: Delete a few elements from $S$ for every large prime, by the earlier proposition.

- In all, delete $O(x/\log x)$ elements from the original $S$
- $\sum_{n \in S} \frac{1}{n} \lesssim 1 - 22 \log \log x/\log x$

Small $q$: Finally, add at most 1 element to $S$ for every small prime, as in the previous slide.

- Final cardinality of $S$ is $\gtrsim (1 - \frac{1}{e})x + O(x \log \log x/\log x)$
- $0 < \sum_{n \in S} \frac{1}{n} \lesssim (1 - 22 \log \log x/\log x) + 1 < 2$
- Denominator of $\sum_{n \in S} \frac{1}{n}$ is not divisible by any prime

Conclusion: $\sum_{n \in S} \frac{1}{n} = 1!$
The largest denominator

“Impossible integers”
Which integers *can’t* be the largest denominator in an Egyptian fraction for 1?

We’ve already seen “no tiny multiples of huge primes”; so the number of these impossible integers up to $x$ is

\[ \gg \frac{x \log \log x}{\log x}. \]

Erdős and Graham asked:
Does the set of impossible integers have positive density, or even density 1?

It turns out the answer is **no**.
The largest denominator

“Impossible integers”

Which integers can’t be the largest denominator in an Egyptian fraction for 1?

We’ve already seen “no tiny multiples of huge primes”; so the number of these impossible integers up to $x$ is

$$\gg \frac{x \log \log x}{\log x}.$$

Erdős and Graham asked:

Does the set of impossible integers have positive density, or even density 1?

It turns out the answer is no.
The largest denominator

“Impossible integers”

Which integers can’t be the largest denominator in an Egyptian fraction for 1?

We’ve already seen “no tiny multiples of huge primes”; so the number of these impossible integers up to \( x \) is

\[
\gg \frac{x \log \log x}{\log x}.
\]

Erdős and Graham asked:

Does the set of impossible integers have positive density, or even density 1?

It turns out the answer is no.
The largest denominator

“Impossible integers”
Which integers \(can’t\) be the largest denominator in an Egyptian fraction for 1?

We’ve already seen “no tiny multiples of huge primes”; so the number of these impossible integers up to \(x\) is

\[ \sim \frac{x \log \log x}{\log x}. \]

Erdős and Graham asked:
Does the set of impossible integers have positive density, or even density 1?

It turns out the answer is \textbf{no}. 
The largest denominator

**Theorem (M., 2000)**

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\ll x \log \log x / \log x$.

**Proof.**

Let $m$ be any integer such that $p | m$ implies $p < m (\log m)^{-22}$. The previous construction works for the rational number $r = 1 - \frac{1}{m}$, since the initial set $S$ of all integers between $\frac{m}{e}$ and $m - 1$ that are not divisible by a prime larger than $m/(\log m)^{22}$ contains all prime factors of the denominator of $r$.

**Conjecture**

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\sim x \log \log x / \log x$. 

Dense Egyptian fractions
The largest denominator

**Theorem (M., 2000)**

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\ll x \log \log x / \log x$.

**Proof.**

Let $m$ be any integer such that $p \mid m$ implies $p < m(\log m)^{-22}$. The previous construction works for the rational number $r = 1 - \frac{1}{m}$, since the initial set $S$ of all integers between $\frac{m}{e}$ and $m - 1$ that are not divisible by a prime larger than $m/(\log m)^{22}$ contains all prime factors of the denominator of $r$.

**Conjecture**

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\sim x \log \log x / \log x$. 
The largest denominator

Theorem (M., 2000)

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\ll x \log \log x / \log x$.

Proof.

Let $m$ be any integer such that $p | m$ implies $p < m (\log m)^{-22}$. The previous construction works for the rational number $r = 1 - \frac{1}{m}$, since the initial set $S$ of all integers between $\frac{m}{e}$ and $m - 1$ that are not divisible by a prime larger than $m/(\log m)^{22}$ contains all prime factors of the denominator of $r$.

Conjecture

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\sim x \log \log x / \log x$. 

Dense Egyptian fractions

Greg Martin
The largest denominator

**Theorem (M., 2000)**

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\ll x \log \log x / \log x$.

**Proof.**

Let $m$ be any integer such that $p | m$ implies $p < m (\log m)^{-22}$. The previous construction works for the rational number $r = 1 - \frac{1}{m}$, since the initial set $S$ of all integers between $\frac{m}{e}$ and $m - 1$ that are not divisible by a prime larger than $m/(\log m)^{22}$ contains all prime factors of the denominator of $r$.

**Conjecture**

The number of integers up to $x$ that cannot be the largest denominator in an Egyptian fraction for 1 is $\sim x \log \log x / \log x$. 
The second-largest denominator

The next Erdős–Graham question

Which integers cannot be the second-largest denominator in an Egyptian fraction for 1? Positive density?
The second-largest denominator

The next Erdős–Graham question
Which integers cannot be the second-largest denominator in an Egyptian fraction for 1? Positive density?

Theorem (M., 2000)
All but finitely many positive integers can be the second-largest denominator in an Egyptian fraction for 1.

Proof.
Given a large integer $m$, choose an integer $M \equiv -1 \pmod{m}$ such that $p \mid M$ implies $p < m(\log m)^{-22}$. Then apply the previous construction to $r = 1 - \frac{1}{m} - \frac{1}{Mm} = 1 - \frac{(M+1)/m}{M}$. 

Dense Egyptian fractions
The second-largest denominator

The next Erdős–Graham question
Which integers cannot be the second-largest denominator in an Egyptian fraction for 1? Positive density?

Theorem (M., 2000)
All but finitely many positive integers can be the second-largest denominator in an Egyptian fraction for 1.

Proof.
Given a large integer $m$, choose an integer $M \equiv -1 \pmod{m}$ such that $p | M$ implies $p < m(\log m)^{-22}$. Then apply the previous construction to $r = 1 - \frac{1}{m} - \frac{1}{Mm} = 1 - \frac{(M+1)/m}{M}$. 

Dense Egyptian fractions

Greg Martin
The second-largest denominator

The next Erdős–Graham question

Which integers cannot be the second-largest denominator in an Egyptian fraction for 1? Positive density?

Theorem (M., 2000)

All but finitely many positive integers can be the second-largest denominator in an Egyptian fraction for 1.

The splitting trick immediately implies: for any \( j \geq 2 \), all but finitely many positive integers can be the \( j \)th-largest denominator in an Egyptian fraction for 1.
All of the implicit constants in the above theorems are effectively computable; so in principle, we know enough to settle the following questions:

**Conjecture 1**
If $m \geq 5$, then $m$ can be the second-largest denominator in an Egyptian fraction for 1. (Note that $m = 2$ and $m = 4$ cannot.)

**Conjecture 2**
If $m \geq 2$ and $j \geq 3$, then $m$ can be the $j$th-largest denominator in an Egyptian fraction for 1. (Our methods establish this for $j \geq j_0$, where $j_0$ is some effectively computable constant.)

**Note:** By splitting trickery, Conjecture 1 implies Conjecture 2.
Issues to settle computationally

All of the implicit constants in the above theorems are effectively computable; so in principle, we know enough to settle the following questions:

**Conjecture 1**

If $m \geq 5$, then $m$ can be the second-largest denominator in an Egyptian fraction for 1. (Note that $m = 2$ and $m = 4$ cannot.)

**Conjecture 2**

If $m \geq 2$ and $j \geq 3$, then $m$ can be the $j$th-largest denominator in an Egyptian fraction for 1. (Our methods establish this for $j \geq j_0$, where $j_0$ is some effectively computable constant.)

Note: By splitting trickery, Conjecture 1 implies Conjecture 2.
All of the implicit constants in the above theorems are effectively computable; so in principle, we know enough to settle the following questions:

**Conjecture 1**

If \( m \geq 5 \), then \( m \) can be the second-largest denominator in an Egyptian fraction for 1. (Note that \( m = 2 \) and \( m = 4 \) cannot.)

**Conjecture 2**

If \( m \geq 2 \) and \( j \geq 3 \), then \( m \) can be the \( j \)th-largest denominator in an Egyptian fraction for 1. (Our methods establish this for \( j \geq j_0 \), where \( j_0 \) is some effectively computable constant.)

Note: By splitting trickery, Conjecture 1 implies Conjecture 2.
Issues to settle computationally

All of the implicit constants in the above theorems are effectively computable; so in principle, we know enough to settle the following questions:

**Conjecture 1**

If $m \geq 5$, then $m$ can be the second-largest denominator in an Egyptian fraction for 1. (Note that $m = 2$ and $m = 4$ cannot.)

**Conjecture 2**

If $m \geq 2$ and $j \geq 3$, then $m$ can be the $j$th-largest denominator in an Egyptian fraction for 1. (Our methods establish this for $j \geq j_0$, where $j_0$ is some effectively computable constant.)

**Note:** By splitting trickery, Conjecture 1 implies Conjecture 2.
The end

Relevant papers of mine

- **Dense Egyptian fractions**

- **Denser Egyptian fractions**

These slides