Friable values of polynomials

How often do the values of a polynomial have only small prime factors?

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April 14, 2006
University of South Carolina Number Theory Seminar

notes to be placed on web page:
www.math.ubc.ca/~gerg/talks.html
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Outline

1. Introduction
2. Bounds for friable values of polynomials
3. Conjecture for prime values of polynomials
4. Conjecture for friable values of polynomials
Friable values of polynomials

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   - Friable integers
   - Friable numbers among values of polynomials

2. Bounds for friable values of polynomials

3. Conjecture for prime values of polynomials

4. Conjecture for friable values of polynomials

Summary
**Friable integers**

**Definition**

\[ \psi(x, y) \] is the number of integers up to \( x \) whose prime factors are all at most \( y \):

\[
\psi(x, y) = \#\{n \leq x : p \mid n \implies p \leq y\}
\]

**Asymptotics:** For a large range of \( x \) and \( y \),

\[
\psi(x, y) \sim x \rho \left( \frac{\log x}{\log y} \right),
\]

where \( \rho(u) \) is the “Dickman–de Bruijn rho-function”.

**Interpretation:** A “randomly chosen” integer of size \( X \) has probability \( \rho(u) \) of being \( X^{1/u} \)-friable.

**In this talk:** Think of \( u = \log x / \log y \) as being bounded above, that is, \( y \geq x^\varepsilon \) for some \( \varepsilon > 0 \).
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The Dickman–de Bruijn \( \rho \)-function

**Definition**

\( \rho(u) \) is the unique continuous solution of the differential-difference equation \( u\rho'(u) = -\rho(u - 1) \) for \( u \geq 1 \) that satisfies the initial condition \( \rho(u) = 1 \) for \( 0 \leq u \leq 1 \).

**Example**

For \( 1 \leq u \leq 2 \),

\[
\rho'(u) = -\frac{\rho(u - 1)}{u} = -\frac{1}{u} \quad \implies \quad \rho(u) = C - \log u.
\]

Since \( \rho(u) = 1 \), we have \( \rho(u) = 1 - \log u \) for \( 1 \leq u \leq 2 \).

**Consequence:** Note that \( \rho(u) = \frac{1}{2} \) when \( u = \sqrt{e} \). Therefore the “median size” of the largest prime factor of \( n \) is \( n^{1/\sqrt{e}} \).
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Conjecture for prime values of polynomials

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(Bateman–Horn conjecture)
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Friable numbers among values of polynomials

Definition

\[ \Psi(F; x, y) \] is the number of integers \( n \) up to \( x \) such that all the prime factors of \( F(n) \) are all at most \( y \):

\[ \Psi(F; x, y) = \# \{ 1 \leq n \leq x : p | F(n) \implies p \leq y \} \]

- When \( F(x) \) is a linear polynomial (friable numbers in arithmetic progressions), we have the same asymptotic \( \Psi(F; x, y) \sim \rho \left( \frac{\log x}{\log y} \right) \).

- Knowing the size of \( \Psi(F; x, y) \) has applications to analyzing the running time of modern factoring algorithms (quadratic sieve, number field sieve).

- A basic sort of question in number theory: are two arithmetic properties (in this case, friability and being the value of a polynomial) independent?
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Introduction

1. Friable values of polynomials
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2. Introduction
   - Friable integers
   - Friable values of polynomials

3. Bounds for friable values of polynomials
   - How friable can values of special polynomials be?
   - How friable can values of general polynomials be?
   - Can we have lots of friable values?

4. Conjecture for prime values of polynomials

5. Conjecture for friable values of polynomials

   A uniform version of Hypothesis H
   (Bateman–Horn conjecture)
   Schinzel’s “Hypothesis H”
   Conjecture for friable values of polynomials

   Statement of the conjecture
   Reduction to convenient polynomials
   Translation into prime values of polynomials
   Shepherding the local factors
   Sums of multiplicative functions

Summary
How friable can values of special polynomials be?

For binomials, there’s a nice trick which yields:

**Theorem (Schinzel, 1967)**

*For any nonzero integers $A$ and $B$, any positive integer $d$, and any $\varepsilon > 0$, there are infinitely many numbers $n$ for which $An^d + B$ is $n^\varepsilon$-friable.*

Balog and Wooley (1998), building on an idea of Eggleton and Selfridge, extended this result to products of binomials

$$\prod_{j=1}^{L} (A_j n^{d_j} + B_j).$$
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Proof for an explicit binomial

Example

For any \( \varepsilon > 0 \), there are infinitely many numbers \( n \) for which \( F(n) = 3n^5 + 7 \) is \( n^\varepsilon \)-friable.

Define \( n_k = 3^{8k-1}2^{2k} \). Then

\[
F(n_k) = 3^{5(8k-1)+1}7^{5(2k)} + 7 = -7((-3^47)^{10k-1} - 1)
\]

factors into values of cyclotomic polynomials:

\[
F(n_k) = -7 \prod_{m|(10k-1)} \Phi_m(-3^47).
\]

\( \Phi_m(x) = \prod_{1 \leq r \leq m} (x - e^{2\pi ir/m}) \)

\( \Phi_m \) has integer coefficients and degree \( \phi(m) \)
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Bounds for friable values of polynomials

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Conjecture for prime values of polynomials
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A uniform version of Hypothesis H

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Summary
From the last slide

- \( F(n) = 3n^5 + 7 \)
- \( F(n_k) = -7 \prod_{m \mid (10k-1)} \Phi_m(-3^47) \)
- \( n_k = 3^{8k-1}7^{2k} \)

- Primes dividing \( F(n_k) \) are \( \leq \max_{m \mid (10k-1)} |\Phi_m(-3^47)| \)
- \( \Phi_m(x) \) is roughly \( x^{\phi(m)} \leq x^{\phi(10k-1)} \)
- \( n_k \) is roughly \( (3^47)^{4k} \), but the largest prime factor of \( F(n_k) \) is bounded by roughly \( (3^47)^{\phi(10k-1)} \)
- Infinitely many \( k \) with \( \phi(10k-1)/4k < \varepsilon \)

How many such friable values? \( \gg_{F,\varepsilon} \log x \), for \( n \leq x \)

\( \varepsilon \) can be made quantitative \( n^{c_F}/\log\log\log n \)-friable values
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Polynomial factorizations

Example

The polynomial $F(x + F(x))$ is always divisible by $F(x)$. In particular, if $\deg F = d$, then $F(x + F(x))$ is roughly $x^{d^2}$ yet is automatically roughly $x^{d^2 - d}$-friable.

Mnemonic

\[ x + F(x) \equiv x \pmod{F(x)} \]

Special case:

- If $F(x)$ is quadratic with lead coefficient $a$, then

\[ F(x + F(x)) = F(x) \cdot aF \left( x + \frac{1}{a} \right). \]

- In particular, if $F(x) = x^2 + bx + c$, then

\[ F(x + F(x)) = F(x)F(x + 1). \]
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A refinement of Schinzel

- Idea: use the reciprocal polynomial $x^d F(1/x)$.
- Restrict to $F(x) = x^d + a_2 x^{d-2} + \ldots$ for simplicity.

**Proposition**

Let $h(x)$ be a polynomial such that $xh(x) - 1$ is divisible by $x^d F(1/x)$. Then $F(h(x))$ is divisible by $x^d F(1/x)$. In particular, we can take $\deg h = d - 1$, in which case $F(h(x))$ is roughly $x^{d^2-d}$ yet is automatically roughly $x^{d^2-2d}$-friable.

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$h(x) \equiv 1/x \pmod{F(1/x)}$

**Note:** The proposition isn’t true for $d = 2$, since the leftover “factor” of degree $2^2 - 2 \cdot 2 = 0$ is a constant.
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Recursively use Schinzel’s construction

\( D_m \): an unspecified polynomial of degree \( m \)

Example

\[ \text{deg} \ F(x) = 4. \text{ Use Schinzel’s construction repeatedly:} \]

\[
\begin{align*}
D_{12} &= F(D_3) = D_4 D_8 \\
D_{84} &= F(D_{21}) = D_{28} D_8 D_{48} \\
D_{3984} &= F(D_{987}) = D_{1316} D_{376} D_{48} D_{2208}
\end{align*}
\]

“score” = 8/3

“score” = 16/7

“score” = 736/329

For \( \text{deg} F = 2 \), begin with \( F(D_4) = D_2 D_2 D_4 \).

Specifically,

\[
F(x + F(x) + F(x + F(x))) = F(x) \cdot aF(x + \frac{1}{a}) \cdot D_4.
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For \( \text{deg} F = 3 \), begin with \( F(D_4) = D_3 D_3 D_6 \).
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  F(x + F(x) + F(x + F(x))) = F(x) \cdot aF(x + \frac{1}{a}) \cdot D_4.
  \]

- For \(\text{deg } F = 3\), begin with \(F(D_4) = D_3 D_3 D_6\).
Recursively use Schinzel’s construction

\[ D_m: \text{ an unspecified polynomial of degree } m \]

**Example**

\[
\text{deg } F(x) = 4. \text{ Use Schinzel’s construction repeatedly:}
\]

\[
D_{12} = F(D_3) = D_4D_8 \\
D_{84} = F(D_{21}) = D_{28}D_8D_{48} \\
D_{3984} = F(D_{987}) = D_{1316}D_{376}D_{48}D_{2208}
\]

“For score” = 8/3

“For score” = 16/7

“For score” = 736/329

- For \( \text{deg } F = 2 \), begin with \( F(D_4) = D_2D_2D_4 \).
  Specifically,

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- For \( \text{deg } F = 3 \), begin with \( F(D_4) = D_3D_3D_6 \).
How friable can values of general polynomials be?

- \( d \geq 4 \): define \( s(d) = d \prod_{j=1}^{\infty} \left( 1 - \frac{1}{u_j(d)} \right) \), where
  
  \[
  u_1(d) = d - 1 \quad \text{and} \quad u_{j+1}(d) = u_j(d)^2 - 2
  \]
- \( s(2) = s(4)/4 \) and \( s(3) = s(6)/4 \)

**Theorem**

(Schinzel, 1967) Given a polynomial \( F(x) \) of degree \( d \geq 2 \), there are infinitely many numbers \( n \) for which \( F(n) \) is \( n^{s(d)} \)-friable.

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\( \varepsilon \) is the multiplicative constant from Schinzel’s Hypothesis H.
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Polynomial substitution yields small lower bounds

**Special case**

Given a quadratic polynomial $F(x)$, there are infinitely many numbers $n$ for which $F(n)$ is $n^{0.55902}$-friable.

**Example**

To obtain $n$ for which $F(n)$ is $n^{0.56}$-friable:

$$D_{168} = F(D_{84}) = D_{42}D_{42}D_{28}D_{8}D_{48} \quad \text{“score”} = \frac{4}{7} > 0.56$$

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The counting function of such $n$ is about $x^{1/3948}$.

“Improvement” Balog, M., Wooley can get $x^{2/3948}$ and an analogous improvement for $\deg F = 3$. 
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Friable values of polynomials

Greg Martin

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Can we have lots of friable values?

Our expectation

For any $\varepsilon > 0$, a positive proportion of values $F(n)$ are $n^\varepsilon$-friable.

We know this for:

- linear polynomials (arithmetic progressions)
  - Hildebrand, then Balog and Ruzsa: $F(n) = n(an + b)$, values $n^\varepsilon$-friable for any $\varepsilon > 0$
  - Hildebrand: $F(n) = (n + 1) \cdots (n + L)$, values $n^\beta$-friable for any $\beta > e^{-1/(L-1)}$
  
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Theorem (Dartyge, M., Tenenbaum, 2001)

Let $F(x)$ be any polynomial, let $d$ be the highest degree of any irreducible factor of $F$, and let $F$ have exactly $K$ distinct irreducible factors of degree $d$. Then for any $\varepsilon > 0$, a positive proportion of values $F(n)$ are $n^{d-1/K+\varepsilon}$-friable.

Remark: for friability of level $n^{d-1}$ or higher, only irreducible factors of degree $\geq d$ matter.

Trivial: $n^d$-friable

Can remove the $\varepsilon$ at the cost of the counting function: the number of $n \leq x$ for which $F(n)$ is $n^{d-1/K}$-friable is

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Schinzel’s “Hypothesis H” (Bateman–Horn conjecture)

**Definition**

\[ \pi(F; x) = \# \{ n \leq x : \text{ } f(n) \text{ is prime for each irreducible factor } f \text{ of } F \} \]

**Conjecture:** \( \pi(F; x) \) is asymptotic to \( H(F) \cdot \text{li}(F; x) \), where:

\[
\begin{align*}
\text{li}(F; x) &= \int_{0}^{x} \frac{dt}{\log |F_1(t)| \cdots \log |F_L(t)|} \quad \text{if } \min\{|F_1(t)|, \ldots, |F_L(t)|\} \geq 2
\\
H(F) &= \prod_{p} \left(1 - \frac{1}{p}\right)^{-L} \left(1 - \frac{\sigma(F; p)}{p}\right)
\end{align*}
\]

- \( L \): the number of distinct irreducible factors of \( F \)
- \( \sigma(F; n) \): the number of solutions of \( F(a) \equiv 0 \pmod{n} \)
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Friable values of polynomials

Greg Martin

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A uniform version of Hypothesis H

**Hypothesis UH**

\[ \pi(F; t) - H(F) \operatorname{li}(F; t) \ll_{d,B} 1 + \frac{H(F) t}{(\log t)^{L+1}} \]

uniformly for all polynomials \( F \) of degree \( d \) with \( L \) distinct irreducible factors, each of which has coefficients bounded by \( t^B \) in absolute value.

- \( \operatorname{li}(F; t) \) is asymptotic to \( \frac{t}{(\log t)^L} \) for fixed \( F \)
- For \( d = K = 1 \), equivalent to expected number of primes, in an interval of length \( y = x^\varepsilon \) near \( x \), in an arithmetic progression to a modulus \( q \leq y^{1-\varepsilon} \)
- Don’t really need this strong a uniformity, but rather on average over some funny family to be described later
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4. Conjecture for friable values of polynomials
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   - Reduction to convenient polynomials
   - Translation into prime values of polynomials
   - Shepherding the local factors
   - Sums of multiplicative functions
What would we expect on probabilistic grounds?

Let $F(x) = f_1(x) \cdots f_L(x)$, where $\deg f_j(x) = d_j$. Let $u > 0$.

- $f_j(n)$ is roughly $n^{d_j}$, and integers of that size are $n^{1/u}$-friable with probability $\rho(d_j u)$.

- Are the friabilities of the various factors $f_j(n)$ independent? This would lead to a prediction involving

$$x \prod_{j=1}^{L} \rho(d_j u).$$

- What about local densities depending on the arithmetic of $F$ (as in Hypothesis H)?
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Conjecture for friable values of polynomials

Let $F(x)$ be any polynomial, let $f_1, \ldots, f_L$ be its distinct irreducible factors, and let $d_1, \ldots, d_L$ be their degrees. Then

$$\psi(F; x, x^{1/u}) = x \prod_{j=1}^{L} \rho(d_j u) + O\left(\frac{x}{\log x}\right)$$

for all $0 < u$.

If $F$ irreducible: $\psi(F; x, x^{1/u}) = x \rho(du) + O(x/\log x)$ for $0 < u$.

Remark: Rather more controversial than Hypothesis H.
Conjecture for friable values of polynomials

Theorem (M., 2002)

Assume Hypothesis UH. Let $F(x)$ be any polynomial, let $f_1, \ldots, f_L$ be its distinct irreducible factors, and let $d_1, \ldots, d_L$ be their degrees. Let $d = \max\{d_1, \ldots, d_L\}$, and let $F$ have exactly $K$ distinct irreducible factors of degree $d$. Then

$$\psi(F; x, x^{1/u}) = x \prod_{j=1}^{L} \rho(d_j u) + O\left(\frac{x}{\log x}\right)$$

for all $0 < u < 1/(d - 1/K)$.

If $F$ irreducible: $\psi(F; x, x^{1/u}) = x \rho(du) + O(x/\log x)$ for $0 < u < 1/(d - 1)$.

Trivial: $0 < u < 1/d$.

Reason to talk about more general $K$: There is one part of the argument that causes an additional difficulty when $K > 1$. 

Conjecture for friable values of polynomials

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Assume Hypothesis UH. Let $F(x)$ be any polynomial, let $f_1, \ldots, f_L$ be its distinct irreducible factors, and let $d_1, \ldots, d_L$ be their degrees. Let $d = \max\{d_1, \ldots, d_L\}$, and let $F$ have exactly $K$ distinct irreducible factors of degree $d$. Then

$$\Psi(F; x, x^{1/u}) = x \prod_{j=1}^{L} \rho(d_j u) + O\left(\frac{x}{\log x}\right)$$

for all $0 < u < 1/(d - 1/K)$.

If $F$ irreducible: $\Psi(F; x, x^{1/u}) = x \rho(du) + O(x / \log x)$ for $0 < u < 1/(d - 1)$.

**Trivial:** $0 < u < 1/d$.

**Reason to talk about more general $K$:** There is one part of the argument that causes an additional difficulty when $K > 1$. 
Reduction to convenient polynomials

Without loss of generality, we may assume:

1. \( F(x) \) is the product of distinct irreducible polynomials \( f_1(x), \ldots, f_K(x) \), all of the same degree \( d \).
2. \( F(x) \) takes at least one nonzero value modulo every prime.
3. No two distinct irreducible factors \( f_i(x), f_j(x) \) of \( F(x) \) have a common zero modulo any prime.

- (1) is acceptable since the friability level exceeds \( x^{d-1} \).
- (2) is not a necessary condition to have friable values (as it is to have prime values). Nevertheless, we can still reduce to this case.
- Both (2) and (3) are achieved by looking at values of \( F(x) \) on suitable arithmetic progressions \( F(Qx + R) \) separately.
Reduction to convenient polynomials

Without loss of generality, we may assume:

1. $F(x)$ is the product of distinct irreducible polynomials $f_1(x), \ldots, f_K(x)$, all of the same degree $d$.
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Under (1), we want to prove that

$$\psi(F; x, x^{1/u}) = x\rho(du)^K + O\left(\frac{x}{\log x}\right)$$

for all $0 < u < 1/(d - 1/K)$. 
Inclusion-exclusion on irreducible factors

**Proposition**

Let $F$ be a primitive polynomial, and let $F_1, \ldots, F_K$ denote the distinct irreducible factors of $F$. Then for $x \geq y \geq 1$,

$$
\Psi(F; x, y) = \lfloor x \rfloor + \sum_{1 \leq k \leq K} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq K} M(F_{i_1} \ldots F_{i_k}; x, y).
$$

**Definition**

$$
M(f; x, y) = \#\{1 \leq n \leq x : \text{for each irreducible factor } g \text{ of } f, \text{ there exists a prime } p > y \text{ such that } p \mid g(n)\}.
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Inclusion-exclusion on irreducible factors

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If we knew that $M(F_{i_1} \cdots F_{i_k}; x, x^{1/u}) \sim x(\log du)^k$, then

$$\Psi(F; x, x^{1/u}) \sim x + \sum_{1 \leq k \leq K} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq K} x(\log du)^k$$

$$= x \left( 1 + \sum_{1 \leq k \leq K} \binom{K}{k} (\log du)^k \right)$$

$$= x(1 - \log du)^K = x \rho(du)^K.$$
Proposition

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If we knew that \( M(F_{i_1} \ldots F_{i_k}; x, x^{1/u}) \sim x(\log du)^k \), then

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\Psi(F; x, x^{1/u}) \sim x + \sum_{1 \leq k \leq K} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq K} x(\log du)^k
\]

\[
= x \left( 1 + \sum_{1 \leq k \leq K} \binom{K}{k} (-\log du)^k \right)
\]

\[
= x(1 - \log du)^K = x \rho(du)^K.
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**Proposition**

Let $F$ be a primitive polynomial, and let $F_1, \ldots, F_K$ denote the distinct irreducible factors of $F$. Then for $x \geq y \geq 1$,

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**Definition**

$$
M(f; x, y) = \#\{1 \leq n \leq x : \text{for each irreducible factor } g \text{ of } f, \text{ there exists a prime } p > y \text{ such that } p \mid g(n)\}.
$$

We want to prove $M(F_{i_1} \ldots F_{i_k}; x, x^{1/u}) \sim x(\log du)^k$. To do this, we sort by the values $n_j = F_{i_j}(n)/p_j$, among those $n$ counted by $M(F_{i_1} \ldots F_{i_k}; x, x^{1/u})$. 
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Proposition

For \( f = f_1 \ldots f_k \) and \( x \) and \( y \) sufficiently large,

\[
M(f; x, y) = \sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} \left( \frac{x - b}{n_1 \cdots n_k} \right) - \pi\left( f_{n_1 \cdots n_k}, b, \eta n_1, \ldots, n_k \right).
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\]

DON’T PANIC
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\]

not important

\[ \xi_j = f_j(x) \approx x^d \]
For \( f = f_1 \ldots f_k \) and \( x \) and \( y \) sufficiently large,

\[
M(f; x, y) = \sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \sum_{b \in \mathcal{R}(f;n_1,\ldots,n_k)} \sum_{(n_i,n_j)=1 \ (1 \leq i < j \leq k)} \left( \pi \left( f_{n_1 \ldots n_k}, b; \frac{x - b}{n_1 \cdots n_k} \right) - \pi \left( f_{n_1 \ldots n_k}, b; \eta n_1,\ldots,n_k \right) \right).
\]

It’s here only because the large primes dividing \( f_j(n) \) had to exceed \( y \). (Later we’ll take \( y = x^{1/u} \).)
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\]

fairly important

\[
\mathcal{R}(f; n_1, \ldots, n_k) = \{ b \ (\text{mod} \ n_1 \cdots n_k) : n_1 \mid f_1(b), n_2 \mid f_2(b), \ldots, n_k \mid f_k(b) \}.
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rather important

\[
f_{n_1 \ldots n_k, b}(t) = \frac{f(n_1 \cdots n_k t + b)}{n_1 \cdots n_k} \in \mathbb{Z}[x]
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In fact, a good understanding of the family \( f_{n_1 \ldots n_k, b} \) is necessary even to treat error terms. However, we’ll only include the details when treating the main term.
**Proposition**

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\]

First: concentrate on

\[
\pi\left(f_{n_1 \ldots n_k}, b; \frac{x - b}{n_1 \cdots n_k}\right) - \pi\left(f_{n_1 \ldots n_k}, b; \eta n_1, \ldots, n_k\right)
\]
Understanding $M(f; x, y)$ inside out

- Look at $\pi\left( n_1 \cdots n_k, b; \frac{x - b}{n_1 \cdots n_k} \right) - \pi\left( n_1 \cdots n_k, b; \eta n_1, \ldots, n_k \right)$

- Upper bound sieve (Brun, Selberg):
  
  $$\pi\left( f_{n_1 \cdots n_k}, \frac{x - b}{n_1 \cdots n_k} \right) + O\left( \frac{H(f_{n_1 \cdots n_k}, b)x}{n_1 \cdots n_k} \frac{(\log x)^{k+1}}{n_1 \cdots n_k} \right)$$

- Main term for $\pi(f; x)$ (we use Hypothesis UH here!):
  
  $$H(f_{n_1 \cdots n_k}, b) \text{li}\left( f_{n_1 \cdots n_k}, \frac{x - b}{n_1 \cdots n_k} \right) + O\left( \frac{H(f_{n_1 \cdots n_k}, b)x}{n_1 \cdots n_k} (\log x)^{k+1} \right)$$

- li is a pretty smooth function:
  
  $$\frac{H(f_{n_1 \cdots n_k}, b)x}{n_1 \cdots n_k} \frac{\log(\xi_1 / n_1) \cdots \log(\xi_k / n_k)}{n_1 \cdots n_k} + O\left( \frac{H(f_{n_1 \cdots n_k}, b)x}{n_1 \cdots n_k} (\log x)^{k+1} \right)$$
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Understanding $M(f; x, y)$ inside out

- Look at $\pi\left(\frac{x - b}{n_1 \cdots n_k}\right) - \pi\left(\frac{x - \eta n_1, \ldots, n_k}{n_1 \cdots n_k}\right)$

- Upper bound sieve (Brun, Selberg):

$$\pi\left(\frac{x - b}{n_1 \cdots n_k}\right) + O\left(\frac{H(f_{n_1 \cdots n_k, b})x}{\log(x)^{k+1}}\right)$$

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Understanding $M(f; x, y)$ inside out

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Understanding $M(f; x, y)$ inside out

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Understanding $M(f; x, y)$ inside out

For $f = f_1 \ldots f_k$ and $x$ and $y$ sufficiently large,

\[
M(f; x, y) = \sum_{n_1 \leq \xi_1/y} \ldots \sum_{n_k \leq \xi_k/y} \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} \left( \pi\left( f_{n_1} \ldots f_{n_k}, b; \frac{x - b}{n_1 \ldots n_k} \right) - \pi\left( f_{n_1} \ldots f_{n_k}, b; n_1 \ldots n_k \right) \right) \\
= \sum_{n_1 \leq \xi_1/y} \ldots \sum_{n_k \leq \xi_k/y} \left( \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_{n_1} \ldots f_{n_k}, b) \right) \\
\times \frac{x}{n_1 \ldots n_k} \cdot \frac{1}{\log(\xi_1/n_1) \ldots \log(\xi_k/n_k)} \left( 1 + O\left( \frac{1}{\log x} \right) \right).
\]

Now we have:

\[
\frac{H(f_{n_1} \ldots n_k, b) x}{\log(\xi_1/n_1) \ldots \log(\xi_k/n_k)} + O\left( \frac{H(f_{n_1} \ldots n_k, b) x}{n_1 \ldots n_k (\log x)^{k+1}} \right)
\]
For $f = f_1 \ldots f_k$ and $x$ and $y$ sufficiently large,

$$M(f; x, y) = \sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} \text{suchthat} \quad (n_i, n_j) = 1 \; (1 \leq i < j \leq k)$$

$$\left( \pi \left( \frac{x - b}{n_1 \cdots n_k} \right) - \pi \left( \frac{f_n_1 \cdots n_k, b; \eta n_1, \ldots, n_k}{n_1 \cdots n_k} \right) \right)$$

$$= \sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \left( \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_n_1 \cdots n_k, b) \right)$$

$$\times \frac{x/n_1 \cdots n_k}{\log(\xi_1/n_1) \cdots \log(\xi_k/n_k)} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Next: concentrate on

$$\sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_n_1 \cdots n_k, b)$$
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Nice sums over local solutions

\[ H(f) = \prod_{p} \left( 1 - \frac{1}{p} \right)^{-k} \left( 1 - \frac{\sigma(f; p)}{p} \right) \]

Recall

\[ \sigma(f; p) = \{ a \pmod{p} : f(a) \equiv 0 \pmod{p} \} \]

Recall

\[ \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_{n_1 \cdots n_k}, b) = H(f)g_1(n_1) \cdots g_k(n_k), \text{ where} \]

\[ g_j(n_j) = \prod_{p^\nu || n_j} \left( 1 - \frac{\sigma(f; p)}{p} \right)^{-1} \left( \frac{\sigma(f_j; p^\nu)}{p} - \frac{\sigma(f_j; p^{\nu+1})}{p} \right). \]

Proposition
Nice sums over local solutions

Recall

\[ H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f; p)}{p}\right) \]

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Proposition

\[ \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_{n_1 \cdots n_k, b}) = H(f)g_1(n_1) \cdots g_k(n_k), \text{ where} \]

\[ g_j(n_j) = \prod_{p^\nu \mid n_j} \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\sigma(f_j; p^\nu) - \frac{\sigma(f_j; p^{\nu+1})}{p}\right). \]
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Statement of the conjecture
Reduction to convenient polynomials
Translation into prime values of polynomials
Shepherding the local factors
Sums of multiplicative functions

Summary

Nice sums over local solutions

Recall

\[ H(f) = \prod_{p} \left( 1 - \frac{1}{p} \right)^{-k} \left( 1 - \frac{\sigma(f; p)}{p} \right) \]

Recall

\[ \mathcal{R}(f; n_1, \ldots, n_k) = \{ b \pmod{n_1 \cdots n_k} : n_1 \mid f_1(b), n_2 \mid f_2(b), \ldots, n_k \mid f_k(b) \} \]

Proposition

\[
\sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_{n_1 \cdots n_k} b) = H(f) g_1(n_1) \cdots g_k(n_k), \text{ where}
\]

\[
g_j(n) = \prod_{p^\nu \mid n} \left( 1 - \frac{\sigma(f; p)}{p} \right)^{-1} \left( \sigma(f_j; p^\nu) - \frac{\sigma(f_j; p^{\nu+1})}{p} \right).
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\sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_{n_1 \cdots n_k}, b) = H(f) g_1(n_1) \cdots g_k(n_k), \\
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Friable values of polynomials

Greg Martin

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\[ H(f) = \prod_p \left( 1 - \frac{1}{p} \right)^{-k} \left( 1 - \frac{\sigma(f; p)}{p} \right) \]

Proving this proposition . . .

. . . is fun, actually, involving the Chinese remainder theorem, counting lifts of local solutions (Hensel’s lemma), and so on.

Proposition

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\sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k) \atop b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_{n_1 \cdots n_k}, b) = H(f) g_1(n_1) \cdots g_k(n_k), \text{ where}
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\[
g_j(n_j) = \prod_{p^\nu \mid n_j} \left( 1 - \frac{\sigma(f; p)}{p} \right)^{-1} \left( \sigma(f_j; p^\nu) - \frac{\sigma(f_j; p^{\nu+1})}{p} \right).
\]
For $f = f_1 \ldots f_k$ and $x$ and $y$ sufficiently large,

$$M(f; x, y) = \sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \left( \sum_{b \in \mathcal{R}(f; n_1, \ldots, n_k)} H(f_{n_1} \ldots n_k, b) \right)$$

$$\times \frac{x/n_1 \cdots n_k}{\log(\xi_1/n_1) \cdots \log(\xi_k/n_k)} \left( 1 + O\left( \frac{1}{\log x} \right) \right)$$

$$= xH(f) \left( 1 + O\left( \frac{1}{\log x} \right) \right)$$

$$\times \sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \frac{g_1(n_1) \cdots g_k(n_k)/n_1 \cdots n_k}{\log(\xi_1/n_1) \cdots \log(\xi_k/n_k)}. \tag{A}$$

Therefore: consider

$$\sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \frac{g_1(n_1) \cdots g_k(n_k)}{n_1 \cdots n_k}$$

$$(n_i, n_j) = 1 \quad (1 \leq i < j \leq k)$$

(take care of logarithms later, via partial summation)
For \( f = f_1 \ldots f_k \) and \( x \) and \( y \) sufficiently large,

\[
M(f; x, y) = \sum_{n_1 \leq \xi_1/y} \ldots \sum_{n_k \leq \xi_k/y} \left( \sum_{b \in \mathcal{R}(f;n_1,\ldots,n_k)} H(f_{n_1\ldots n_k},b) \right) \times \frac{x/n_1 \ldots n_k}{\log(\xi_1/n_1) \ldots \log(\xi_k/n_k)} \left(1 + O\left(\frac{1}{\log x}\right)\right) \
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\((n_i,n_j)=1 \ (1 \leq i < j \leq k)\)

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For $f = f_1 \ldots f_k$ and $x$ and $y$ sufficiently large,

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First: consider more general sums of multiplicative functions.
Multiplicative functions: one-variable sums

Definition
Let’s say a multiplicative function \( g(n) \) is \( \alpha \) on average if it takes nonnegative values and

\[
\sum_{p \leq w} \frac{g(p) \log p}{p} \sim \alpha \log w.
\]

Note: we really need upper bounds on \( g(p^\nu) \) as well . . .

Lemma
If the multiplicative function \( g(n) \) is \( \alpha \) on average, then

\[
\sum_{n \leq t} \frac{g(n)}{n} \sim c(g)(\log t)^\alpha,
\]

where \( c(g) = \prod_p \left(1 - \frac{1}{p}\right)^\alpha \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots \right). \)
Multiplicative functions: one-variable sums

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Let’s say a multiplicative function $g(n)$ is $\alpha$ on average if it takes nonnegative values and

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Multiplicative functions: one-variable sums

Greg Martin
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Multiplicative functions: more variables

From previous slide

\[ c(g) = \prod_p \left( 1 - \frac{1}{p} \right)^{\alpha} \left( 1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots \right) \]

By the lemma on the previous slide, we easily get:

Proposition

*If the multiplicative functions \( g_1(n), \ldots, g_k(n) \) are each 1 on average, then*

\[
\sum_{n_1 \leq t} \cdots \sum_{n_k \leq t} \frac{g_1(n_1) \cdots g_k(n_k)}{n_1 \cdots n_k} \sim c(g_1) \cdots c(g_k)(\log t)^k.
\]

However, we need the analogous sum with the coprimality condition \((n_i, n_j) = 1\). (This is where \( K > 1 \) makes life harder!)
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However, we need the analogous sum with the coprimality condition \((n_i, n_j) = 1\). (This is where \(K > 1\) makes life harder!)

Never mind that \(g_1 + \cdots + g_k\) isn’t multiplicative!
Partial summation: return of the logs

The proposition on the previous slide:

\[ \sum_{n_1 \leq \frac{\xi_1}{y}} \cdots \sum_{n_k \leq \frac{\xi_k}{y}} \frac{g_1(n_1) \cdots g_k(n_k)}{n_1 \cdots n_k} \]

\[ \sim c(g_1 + \cdots + g_k) \prod_{j=1}^{k} \log \frac{\xi_j}{y}. \]

For our functions, \( g_j(p) = \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\sigma(f_j; p) - \frac{\sigma(f_j; p^2)}{p}\right) \)

\[ = \sigma(f_j; p)(1 + O\left(\frac{1}{p}\right)), \] and \( \sigma(f_j; p) \) is indeed 1 on average by the prime ideal theorem.
Partial summation: return of the logs

The proposition on the previous slide . . .
. . . gives, after a $k$-fold partial summation argument:

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*If the multiplicative functions $g_1(n), \ldots, g_k(n)$ are each 1 on average, then*

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\sum_{n_1 \leq \xi_1/y} \cdots \sum_{n_k \leq \xi_k/y} \frac{g_1(n_1) \cdot \cdots \cdot g_k(n_k)}{n_1 \cdots n_k \log(\xi_1/n_1) \cdots \log(\xi_k/n_k)} \sim c(g_1 + \cdots + g_k) \prod_{j=1}^{k} \log \frac{\xi_j}{\log y}.
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For our functions, $g_j(p) = (1 - \frac{\sigma(f_j; p)}{p})^{-1} (\sigma(f_j; p) - \frac{\sigma(f_j; p^2)}{p}) = \sigma(f_j; p)(1 + O(\frac{1}{p}))$, and $\sigma(f_j; p)$ is indeed 1 on average by the prime ideal theorem.
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$$

$$
\sim c(g_1 + \cdots + g_k) \prod_{j=1}^k \log \frac{\log \xi_j}{\log y}.
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= \sigma(f_j; p)(1 + O(\frac{1}{p})), \text{ and } \sigma(f_j; p) \text{ is indeed 1 on average by the prime ideal theorem.}
$$
For \( f = f_1 \ldots f_k \) and \( x \) and \( y \) sufficiently large,

\[
M(f; x, y) = xH(f) \left( 1 + O \left( \frac{1}{\log x} \right) \right) \\
\times \sum_{n_1 \leq \xi_1 / y} \cdots \sum_{n_k \leq \xi_k / y} \frac{g_1(n_1) \cdots g_k(n_k)/n_1 \cdots n_k}{\log(\xi_1/n_1) \cdots \log(\xi_k/n_k)}
\]

\[
= H(f)c(g_1 + \cdots + g_k) \\
\times x \left( \prod_{j=1}^{k} \frac{\log \xi_j}{\log y} \right) \left( 1 + O \left( \frac{1}{\log x} \right) \right).
\]

Recall: \( \xi_j = f_j(x) \approx x^d \), and we care about \( y = x^{1/u} \). Then \( \log(\log \xi_j / \log y) \sim \log du \).

We have the order of magnitude \( x(\log du)^k \) we wanted . . . but what about the local factors \( H(f)c(g_1 + \cdots + g_k) \)?
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\[
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The magic moment for $H(f)c(g_1 + \cdots + g_k)$

- $H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f; p)}{p}\right)$
- $c(g) = \prod_p \left(1 - \frac{1}{p}\right)^\alpha \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)$

We have $g_j(p^\nu) = \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\sigma(f_j; p^\nu) - \frac{\sigma(f_j; p^{\nu+1})}{p}\right)$,

and so

\[
\frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu} = \frac{1}{p^\nu} \sum_{j=1}^k \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f_j; p^\nu)}{p^\nu} - \frac{\sigma(f_j; p^{\nu+1})}{p^{\nu+1}}\right) = \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f; p^\nu)}{p^\nu} - \frac{\sigma(f; p^{\nu+1})}{p^{\nu+1}}\right)
\]

since the $f_j$ have no common roots modulo $p$. 
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and so $\frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$

$$= \frac{1}{p^\nu} \sum_{j=1}^k \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f_j; p^\nu)}{p^\nu} - \frac{\sigma(f_j; p^{\nu+1})}{p^{\nu+1}}\right)$$

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We have $g_j(p^\nu) = \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\sigma(f_j; p^\nu) - \frac{\sigma(f_j; p^{\nu+1})}{p}\right)$,
and so $\frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$

$$= \frac{1}{p^\nu} \sum_{j=1}^{k} \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f_j; p^\nu)}{p^\nu} - \frac{\sigma(f_j; p^{\nu+1})}{p^{\nu+1}}\right)$$

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Therefore

$$1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^n)}{p^n}$$

$$= 1 + \sum_{\nu=1}^{\infty} \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f; p^{\nu})}{p^{\nu}} - \frac{\sigma(f; p^{\nu+1})}{p^{\nu+1}}\right)$$
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- $H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f; p)}{p}\right)$
- $c(g) = \prod_p \left(1 - \frac{1}{p}\right)^\alpha \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)$

Therefore

$$1 + \sum_{\nu=1}^\infty \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$$

$$= 1 + \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \sum_{\nu=1}^\infty \left(\frac{\sigma(f; p^\nu)}{p^\nu} - \frac{\sigma(f; p^{\nu+1})}{p^{\nu+1}}\right)$$

This is a telescoping sum . . .
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Therefore

\[
\begin{align*}
1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu} &= 1 + \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f; p)}{p}\right) \\
\text{This is a telescoping sum . . . tada!}
\end{align*}
\]
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Therefore

$$1 + \sum_{\nu=1}^\infty \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}$$

$$= 1 + \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f; p)}{p}\right) =$$

And this whole expression simplifies . . .
The magic moment for $H(f)c(g_1 + \cdots + g_k)$

- $H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f; p)}{p}\right)$
- $c(g) = \prod_p \left(1 - \frac{1}{p}\right)^\alpha \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)$

Therefore

\[
1 + \sum_{\nu=1}^\infty \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}
\]

\[
= 1 + \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} \left(\frac{\sigma(f; p)}{p}\right) = \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1}.
\]

And this whole expression simplifies \ldots nicely.
Friable values of polynomials

Introduction
Friable integers
Friable values of polynomials

Bounds for friable values of polynomials
How friable can values of special polynomials be?
How friable can values of general polynomials be?
Can we have lots of friable values?

Conjecture for prime values of polynomials
Schinzel’s “Hypothesis H”
(Bateman–Horn conjecture)
A uniform version of Hypothesis H

Conjecture for friable values of polynomials
Statement of the conjecture
Reduction to convenient polynomials
Translation into prime values of polynomials
Shepherding the local factors
Sums of multiplicative functions

Summary
The magic moment for $H(f)c(g_1 + \cdots + g_k)$

- $H(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\sigma(f; p)}{p}\right)$
- $c(g) = \prod_p \left(1 - \frac{1}{p}\right)^{\alpha} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)$

We conclude that

$$H(f)c(g_1 + \cdots + g_k)$$

$$= H(f) \prod_p \left(1 - \frac{1}{p}\right)^{k} \left(1 + \sum_{\nu=1}^{\infty} \frac{(g_1 + \cdots + g_k)(p^\nu)}{p^\nu}\right)$$

$$= H(f) \prod_p \left(1 - \frac{1}{p}\right)^{k} \left(1 - \frac{\sigma(f; p)}{p}\right)^{-1} = 1$$

\[\cdots \text{ amazing!}\]
There are **lots of open problems** concerning friable values of polynomials—and many possible improvements from a single clever new idea.

The **asymptotics** for friable values of polynomials depends on the degrees of their irreducible factors—but shouldn’t depend on the polynomial otherwise.

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**Notes to be placed on web page**

www.math.ubc.ca/~gerg/talks.html