10/03/2019

Lecture 10

Recap: Last time: Finite difference ~~equations~~ approximation to differential equations.

BVP:

$$\begin{cases} f''(x) + q(x) f(x) = r(x), & x \in [0,1] \\ f(0) = A \\ f(1) = B. \end{cases}$$

①. Discretize $f(x)$ on $0 = x_0 < x_1 < \dots < x_N = 1$

$x_j = x_{j-1} + \Delta x.$

Get

$$F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \in \mathbb{R}^{N+1}$$

②. ~~Express f''~~ Discretize $f''(x)$ at interior points $x_1, \dots, x_{N-1}$

$$\begin{bmatrix} f''(x_1) \\ \vdots \\ f''(x_{N-1}) \end{bmatrix} \approx F'' = \begin{bmatrix} F_1'' \\ F_2'' \\ \vdots \\ F_{N-1}'' \end{bmatrix} \in \mathbb{R}^{N-1}$$

Express F'' in terms of F:

$$F'' = \frac{1}{(\Delta x)^2} D_{N-1} \cdot D_N F, \quad \text{where } D_N = \underbrace{\begin{bmatrix} -1 & 1 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & & & & -1 & 1 \end{bmatrix}}_{N \times (N+1)}$$

and

$$D_{N-1} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \\ \end{bmatrix} \quad (N-1) \times N$$

LHS: $f''(x) + q(x) f(x)$

discretization gives:

$$F'' + \begin{bmatrix} q_1 f_1 \\ q_2 f_2 \\ \vdots \\ q_{N-1} f_{N-1} \end{bmatrix} = \Bigg(\underset{(N-1)\times(N+1)}{\frac{1}{(\Delta x)^2} D_N D_{N-1}} + \underset{(N-1)\times(N+1)}{\overbrace{\begin{bmatrix} 0 & q_1 & 0 & \cdots & 0 \\ 0 & 0 & q_2 & \cdots & 0 \\ & & & \ddots & \vdots \\ 0 & \cdots & 0 & q_{N-1} & 0 \end{bmatrix}}^{:= Q}} \Bigg) \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}$$

$$= \left(\frac{1}{(\Delta x)^2} D_N D_{N-1} + Q \right) F$$

RHS: $r(x)$, discretize to get $R = \begin{bmatrix} r(x_1) \\ \vdots \\ r(x_{N-1}) \end{bmatrix}$

Set them equal, and add in Boundary Values:

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ - & \frac{1}{(\Delta x)^2} D_N D_{N-1} + Q & & - \\ 0 & 0 & \cdots & 0 \; 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}}_{F} = \underbrace{\begin{bmatrix} A \\ | \\ R \\ | \\ B \end{bmatrix}}_{b}.$$

L, b known, need to find F.

MATLAB.

Chapter II. Subspaces, ~~[illegible]~~, Bases, Dimension

II.1. Vectors spaces and subspaces

What is a vector?

- an n-tuple is a vector

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad x_j \in \mathbb{R}$$ (has a direction and a magnitude)

- but also, a polynomial is a vector! Or a function on $[0,1]$!

What do we need?

- a set of vectors V (e.g. $\mathbb{R}^n$)
- vector addition in V (standard component-wise addition in $\mathbb{R}$
- a set of scalars F (for $V = \mathbb{R}^n$, $F = \mathbb{R}$)
- addition/multiplication of scalars (standard addition/mult. in $\mathbb{R}$
- multiplication of a vector by a scalar.

Definition: V is a vector space over the scalars F if $\forall u, v \in V$

(1) $(V, +)$ is a "commutative group", i.e. it satisfies

(a). ~~[illegible]~~ $u + v \in V$

(b). $u + v = v + u$ commutativity

(c). $u + (v + w) = (u + v) + w$ associativity

(d). $\exists \vec{0}$ s.t. $u + \vec{0} = u$ $\forall u \in V$.

(e) $\exists -u$ s.t. $u + (-u) = \vec{0}$.

(2). For scalars $a, b \in F$, $u, v \in V$

(a) $au \in V$

(b) $a(u+v) = au + av$

(c) $(a+b)u = au + bu$

(d) $1 \cdot u = u$.

Examples: (1) $V = \mathbb{R}^n$, $F = \mathbb{R}$

(2) V = set of all $n \times n$ matrices, $F = \mathbb{R}$ with

$$sM = [sM_{ij}]_{n \times n}$$

(3) V = set of all real-valued functions on $[0,1]$, $F = \mathbb{R}$.

e.g. $f(x) = e^x \sin x$ on $[0,1]$ is a "vector" on V.

(4). V = set of all continuous functions on $\mathbb{R}$.

(5). $V = [0,1]$ is <u>not</u> a vector space!

- no additive invers of $x = 0.5$ in V
  $-0.5 \notin V$

- not closed under addition:
  $x = 0.6$, $y = 0.8$
  $x+y = 1.4 \notin V$.

Vector subspaces

Definition: Let V be a vector space. A subset $S \subseteq V$ is a subspace of V if $\forall u, v \in S$ and $\forall a, b$ scalars:

$$\left.\begin{array}{l}(1)\ u+v \in S \\ (2)\ au \in S\end{array}\right\} \Leftrightarrow (1')\ au+bv \in S$$

Examples: 1. $S = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\}$ ← xz-plane!

Claim: S is a subspace of $V = \mathbb{R}^3$.

Proof: Let $u = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} \in S$ and let $a, b \in \mathbb{R}$.

let's check (1') $au + bv = \begin{bmatrix} ax_1 + bx_2 \\ 0 \\ az_1 + bz_2 \end{bmatrix} \in S \Rightarrow (1')$ holds
$\Rightarrow S$ is a subspace of $\mathbb{R}^3$.