~~004~~ Lecture 8   10/01/2019   ⊛ Quiz 2 next time

Finite Difference Approximations (I.3)

Goal: Find approximate numerical solution of certain differential equations.

Specifically, we'll focus on $f''(x) + q(x) f(x) = r(x)$,

where $q(x)$ and $r(x)$ are known functions.

$f(x)$ is unknown and will be approximated.

Example: $f''(x) = 4 \quad (*)$ fits this format with $q(x)=0$ and $r(x)=4$.

To solve exactly,

$$f'(x) = 4x + c_1, \quad c_1 \in \mathbb{R} \text{ arbitrary constant}$$

$$\Rightarrow f''(x) = 2x^2 + c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R} \text{ arbitrary constants}$$

This $f(x)$ is the general solution of the DE $(*)$.

To get a unique solution, we need to impose additional constraints:

Possibility 1: Specify the "initial values" of $f(x)$ and $f'(x)$, e.g.

$$f(0) = 1, \quad f'(0) = 2.$$

Problem:
$$\begin{cases} f''(x) = 4 \\ f(0) = 1 \\ f'(0) = 2 \end{cases}$$
Initial Value Problem (IVP).

Solution: We already know that $f(x) = 2x^2 + c_1 x + c_2$

To find c_1 and c_2 use initial values:

$f(0) = 1 \Rightarrow c_2 = 1$

$f(x) = 2x^2 + 2x + 1$

Possibility 2: Specify the "boundary values" of f, i.e.,

$$\left.\begin{aligned} f''(x) &= 4 \\ f(0) &= 1 \\ f(1) &= 6 \end{aligned}\right\} \text{boundary value problem (BVP)}$$

Solution: $f(x) = 2x^2 + c_1 x + c_2$

$f(0) = 1 \Rightarrow c_2 = 1$

$f(1) = 6 \Rightarrow 2 + c_1 + c_2 = 6$

$2 + c_1 + 1 = 6 \Rightarrow c_1 = 3$

$\Rightarrow f(x) = 2x^2 + 3x + 1$ solves the BVP.

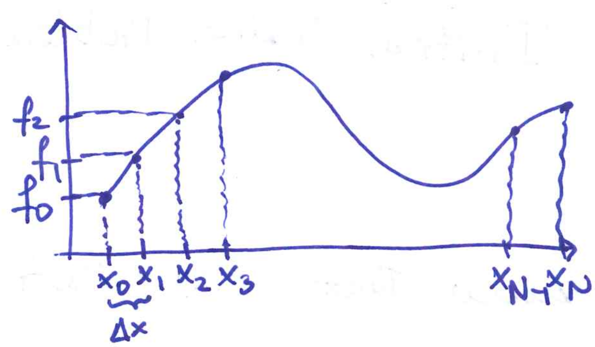
PROBLEM: often, even simple-looking DEs are hard to solve exactly; OR there is NO explicit solution!

Example: $f''(x) + \cos(x) f(x) = x^2$

$\rightsquigarrow$ no explicit solution!

Remedy: Find an approximate numerical solution by discretizing the BVP and turning it into a matrix equation!

HOW?



$$\boxed{\begin{aligned} f_j = f(x_j), \quad & x_j = x_0 + j \cdot \Delta x \\ & j = 0, 1, \ldots, N. \end{aligned}}$$

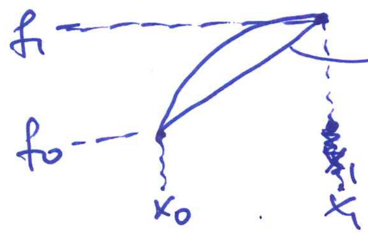
$(N+1)$-pt discretization of f on the interval $[x_0, x_N]$.

Define: $F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}$ discrete $(N+1)$ point approximation of $f(x)$. Note $F \in \mathbb{R}^{N+1}$.

We need:

(1) Approximate the first derivative

$$f'(x_0), f'(x_1), \ldots, f'(x_N)$$



slope of this line is $\approx f'(x_0)$.

That is: $f'(x_j) \approx \dfrac{f_{j+1} - f_j}{\Delta x}$.

Then,

$$F' = \frac{1}{\Delta x} \begin{bmatrix} f_1 - f_0 \\ f_2 - f_1 \\ \vdots \\ f_N - f_{N-1} \end{bmatrix} \approx \begin{bmatrix} f'(x_0) \\ f'(x_1) \\ \vdots \\ f'(x_{N-1}) \end{bmatrix}$$

Observe:

$$F' = \frac{1}{\Delta x} \underbrace{\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 \cdots & 0 \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}}_{D_N \,:\, N \times (N+1) \text{ matrix}} \cdot \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}}_{F}$$

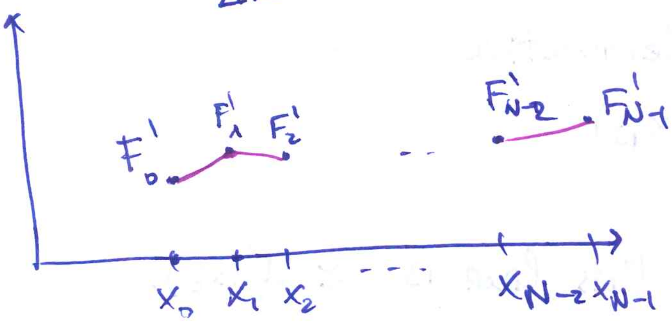
In short: $F' = \frac{1}{\Delta x} D_N \cdot F$.

Next:

$$F'' = \frac{1}{\Delta x}\begin{bmatrix} F_1' - F_0' \\ F_2' - F_1' \\ \vdots \\ F_{N-1}' - F_{N-2}' \end{bmatrix} \approx \begin{bmatrix} f''(x_1) \\ f''(x_2) \\ \vdots \\ f''(x_{N-1}) \end{bmatrix}$$

$$= \frac{1}{\Delta x} D_{N-1} F'$$

Second derivatives at the "interior points".



Let's combine:

$$F'' = \frac{1}{(\Delta x)^2} D_{N-1} D_N F$$

Note that $\underset{(N-1)\times N}{D_{N-1}} \; \underset{N\times(N+1)}{D_N} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & 0 \\ & 0 & & & \ddots & \ddots & \ddots \\ & & & & 1 & -2 & 1 \end{bmatrix}$

$(N-1)\times(N+1)$

Example: Solve the BVP approximately using an $(N+1)$-point discretization.

$$\begin{cases} f''(x) = r(x) \\ f(0) = A \\ f(1) = B \end{cases}$$

Solution: Need:

(a). discretize the LHS of the equation

(b). discretize the RHS of the equation at the same points as the LHS

(c). incorporate two boundary values.

We already know (a):

$$\text{LHS} \longrightarrow F'' = \frac{1}{(\Delta x)^2} D_{N-1} D_N F$$

at $x_1, x_2, \dots, x_{N-1}$ (interior points).

(b) $$R = \begin{bmatrix} r(x_1) \\ \vdots \\ r(x_{N-1}) \end{bmatrix}$$ (remember that $r(x)$ is known)

Thus, at $x_1, x_2, \dots, x_{N-1}$ we have

$$F'' = R \quad \leftarrow \text{discretized DE!}$$

$$(\Leftrightarrow) \quad \underbrace{\frac{1}{(\Delta x)^2} D_{N-1} D_N}_{\text{known}} \overset{\text{unknown}}{F} = \underbrace{R}_{\text{known}}$$

N-1 equations

N+1 unknowns

Need 2 more equations:

(c) $f(0) = f_0 = A$ $(*)$

$f(L) = f_N = B$ $(**)$

$$(*) \Leftrightarrow [1 \; 0 \; \cdots \; 0] \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}}_{F} = A$$

$$(**) \Leftrightarrow [0 \; 0 \; \cdots \; 0 \; 1] F = B$$

Then, the full discretization of the BVP is:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ & & & & \\ - & D_{N-1} D_N & & - & \\ & & & & \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} F = \begin{bmatrix} A \\ | \\ (\Delta x)^2 R \\ | \\ B \end{bmatrix}$$

$\Rightarrow$ The discretized solution F is given by the solution of

$$L \overset{}{F} = b$$

only unknown.

<u>Question:</u> Is L invertible?

<u>Answer:</u> Yes! In fact, $\det(L) = \pm N$

<u>Exercise:</u> Prove this for $N=4$.

<u>Now:</u> bring in $q(x)$:

$$\boxed{\begin{array}{l} f''(x) + q(x) f(x) = r(x) \\ f(0) = A;\ f(1) = B \end{array}}$$ Solve on $[0,1]$

Need to discretize $q(x) f(x)$ as well!

$$\begin{bmatrix} q(x_1) f(x_1) \\ q(x_2) f(x_2) \\ \vdots \\ \\ q(x_{N-1}) f(x_{N-1}) \end{bmatrix}$$

let $q_i = q(x_i)$

Then,

$$\underbrace{\begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 0 & q_1 & 0 & \cdots & 0 \\ \vdots & & q_2 & & \vdots \\ \vdots & & & \ddots & \\ & & & q_{N-1} & 0 \\ 0 & \cdots & & & 0 \end{bmatrix}}_{Q:\ (N+1)\times(N+1)} \overbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ \\ f_{N-1} \\ f_N \end{bmatrix}}^{F}$$

$\gg$ Need to solve

$$\left(L + (\Delta x)^2 Q\right) F = \underbrace{\begin{bmatrix} A \\ (\Delta x)^2 R \\ B \end{bmatrix}}_{b}$$

unknown!

MATLAB.