II. Cubic spline interpolation

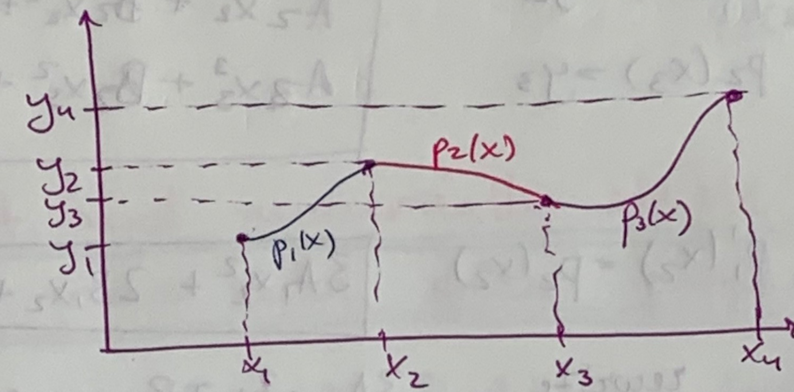
Lecture 7 — 09/26/2019

Given $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Fit a different polynomial

$p_j(x) = A_j x^3 + B_j x^2 + C_j x + D_j$ between x_j and x_{j+1} $\forall j = 1, \ldots, n-1$.

→ cubic polynomial.



$4(n-1)$ unknowns $\{A_j, B_j, C_j, D_j : j = 1, \ldots, n-1\}$.

~~In addition~~, Impose the following conditions:

(C1) Continuity and interpolation: $p_j(x_j) = y_j$, $p_j(x_{j+1}) = y_{j+1}$ $\forall j = 1, \ldots, n-1$ $\boxed{2(n-1) \text{ eqs}}$

(C2). Differentiability: $p_j'(x_{j+1}) = p_{j+1}'(x_{j+1})$ $\forall j = 1, \ldots, n-2$ $\boxed{(n-2) \text{ eqs}}$

(C3). Twice differentiable at $x_2, \ldots, x_{n-1}$: $\boxed{(n-2) \text{ eqs}}$

$$p_j''(x_{j+1}) = p_{j+1}''(x_{j+1}) \quad \forall j = 1, \ldots, n-2.$$

(C4). Impose $f''(x_1) = f''(x_n) = 0$ $\boxed{2 \text{ eqs}}$

$$2(n-1) + (n-2) + (n-2) + 2 = 4(n-1) \text{ equations}$$

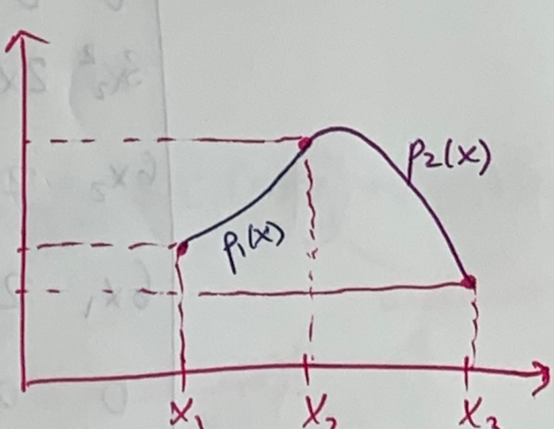
and $4(n-1)$ unknowns.

Example: Set $n = 3$. Let's solve.

$p_1(x) = A_1 x^3 + B_1 x^2 + C_1 x + D_1$

$p_2(x) = A_2 x^3 + B_2 x^2 + C_2 x + D_2$.

$\boxed{8 \text{ unknowns}}$



(C1) $p_1(x_1) = y_1$ $\quad$ $A_1 x_1^3 + B_1 x_1^2 + C_1 x_1 + D_1 = y_1$

$p_1(x_2) = y_2$ $\quad$ $A_1 x_2^3 + B_1 x_2^2 + C_1 x_2 + D_1 = y_2$

$p_2(x_2) = y_2$ $\quad$ $A_2 x_2^3 + B_2 x_2^2 + C_2 x_2 + D_2 = y_2$

$p_2(x_3) = y_3$ $\quad$ $A_2 x_3^3 + B_2 x_3^2 + C_2 x_3 + D_2 = y_3$

4 eqs.

(C2) $p_1'(x_2) = p_2'(x_2)$ $\quad$ $\boxed{3A_1 x_2^2 + 2B_1 x_2 + C_1 = 3A_2 x_2^2 + 2B_2 x_2 + C_2}$ 1 eq

rewrite $\quad 3A_1 x_2^2 + 2B_1 x_2 + C_1 - 3A_2 x_2^2 - 2B_2 x_2 - C_2 = 0.$

(C3) $p_1''(x_2) = p_2''(x_2)$ $\quad$ $\boxed{6A_1 x_2 + 2B_1 = 6A_2 x_2 + 2B_2}$ 1 eq.

rewrite $\quad 6A_1 x_2 + 2B_1 - 6A_2 x_2 - 2B_2 = 0$

(C4) $p_1''(x_1) = 0$ $\quad$ $\boxed{6A_1 x_1 + 2B_1 = 0}$

$p_2''(x_3) = 0$ $\quad$ $\boxed{6A_2 x_3 + 2B_2 = 0}$

2 eq.

$\Rightarrow$ 8 equations

8 unknowns!

Let's write the linear system using matrix notation:

$$\underbrace{\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 \\ x_2^3 & x_2^2 & x_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2^3 & x_2^2 & x_2 & 1 \\ 0 & 0 & 0 & 0 & x_3^3 & x_3^2 & x_3 & 1 \\ 3x_2^2 & 2x_2 & 1 & 0 & -3x_2^2 & -2x_2 & -1 & 0 \\ 6x_2 & 2 & 0 & 0 & -6x_2 & -2 & 0 & 0 \\ 6x_1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6x_3 & 2 & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

Need to solve: $Sa = b$,

where S and b are known, and a is unknown.

Once we find a, we will then get $p_1(x)$ and $p_2(x)$.

$\boxed{\text{MATLAB}}$

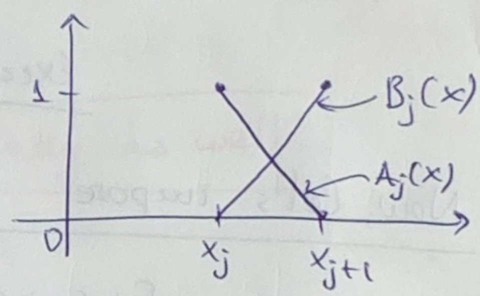
A more efficient and numerically stable method for solving the same cubic spline problem

Problem: Given $(x_1, y_1), \dots, (x_n, y_n)$ find $p_j(x)$ (cubic polynomials) s.t. (C1)-(C4) are satisfied. Same problem as before!

New approach: (I.2.6)

Define: ① $A_j(x) := \dfrac{x_{j+1} - x}{x_{j+1} - x_j}$

$$B_j(x) := 1 - A_j(x) = \frac{x - x_j}{x_{j+1} - x_j}$$



Note: $A_j(x_j) = 1$, $A_j(x_{j+1}) = 0$

② . $C_j(x) := \frac{1}{6}\left(A_j^3(x) - A_j(x)\right)(x_{j+1} - x_j)^3$

$$D_j(x) := \frac{1}{6}\left(B_j^3(x) - B_j(x)\right)(x_{j+1} - x_j)^3$$

Note: A_j, B_j are polynomials of degree 1

C_j, D_j " " " 3

Next: Write $p_j(x) = A_j(x) y_j + B_j(x) y_{j+1} + C_j(x) z_j + D_j(x) z_{j+1}$ unknowns!

$j = 1, \dots, n-1$

The only unknowns are: $z_1, \dots, z_n$ (n unknowns)!

Want: Impose (C1)-(C4) to find the values of $z_1, \dots, z_n$!

Observations:

① $$p_j^{(k)}(x) = \begin{bmatrix} A_j^{(k)}(x) & B_j^{(k)}(x) & C_j^{(k)}(x) & D_j^{(k)}(x) \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix},$$

(where $p_j^{(k)}(x)$ is the k-th derivative of $p_j(x)$ and $p_j^{(0)}(x) = p_j(x)$)

② $A_j(x_j) = 1$, $A_j(x_{j+1}) = 0$, $A_j''(x_j) = 0$, $A_j''(x_{j+1}) = 0$

$B_j(x_j) = 0$, $B_j(x_{j+1}) = 1$, $B_j''(x_j) = 0$, $B_j''(x_{j+1}) = 0$

$C_j(x_j) = 0$, $C_j(x_{j+1}) = 0$, $C_j''(x_j) = 1$, $C_j''(x_{j+1}) = 0$

$D_j(x_j) = 0$, $D_j(x_{j+1}) = 0$, $D_j''(x_j) = 0$, $D_j''(x_{j+1}) = 1$.

Exercise: Check these!

Now, let's impose (C1)-(C4): ① and ② imply

$$p_j(x_j) = \begin{bmatrix} A_j(x_j) & B_j(x_j) & C_j(x_j) & D_j(x_j) \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix}$$

$$\overset{\text{by } ②}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} = y_j$$

So, $p_j(x_j) = y_j$ automatically!

$$p_j(x_{j+1}) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} = y_{j+1}.$$

$\Rightarrow p_j(x_{j+1}) = y_{j+1}$ automatically! $\Rightarrow$ (C1) is satisfied!

Next, check (C3): $p_j''(x_{j+1}) = p_{j+1}''(x_{j+1}), \quad \forall j = 1, \dots, n-2.$

Let's see:

$$p_j''(x_{j+1}) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} = z_{j+1}.$$

On the other hand,

$$p_{j+1}''(x_{j+1}) = \begin{bmatrix} A_{j+1}''(x_{j+1}) & B_{j+1}''(x_{j+1}) & C_{j+1}''(x_{j+1}) & D_{j+1}''(x_{j+1}) \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} = z_{j+1}$$

$\boxed{\Rightarrow p_j''(x_{j+1}) = z_{j+1} = p_{j+1}''(x_{j+1})}$

$\boxed{\Rightarrow \text{(C3) holds automatically as well!}}$

Next, (C4): $p_1''(x_1) = 0$

But $p_1''(x_1) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix} = z_1 \quad \Rightarrow \quad \boxed{z_1 = 0}$

new eqns

Similarly, $p_{n-1}''(x_n) = z_n \quad \Rightarrow \quad \boxed{z_n = 0}$

We still need to determine $z_2, z_3, \dots, z_{n-1}$ from

$\boxed{(C2): \quad p_j'(x_{j+1}) = p_{j+1}'(x_{j+1})}$

After explicit calculation (see notes; also exercise!), we get $n-2$ equations to determine $z_2, z_3, \dots, z_{n-1}$:

$$\left(\frac{x_{j+1}-x_j}{6}\right)z_j + \left(\frac{x_{j+2}-x_j}{3}\right)z_{j+1} + \left(\frac{x_{j+2}-x_{j+1}}{6}\right)z_{j+2} = \frac{y_{j+2}-y_{j+1}}{x_{j+2}-x_{j+1}} - \frac{y_{j+1}-y_j}{x_{j+1}-x_j}$$

for $j = 1, 2, \dots, n-2$.

Using matrix notation:

$$S = \begin{bmatrix} 1 & 0 & 0 & & \cdots & & 0 \\ \frac{x_2-x_1}{6} & \frac{x_3-x_1}{3} & \frac{x_3-x_2}{6} & 0 & \cdots & & 0 \\ 0 & \frac{x_3-x_2}{6} & \frac{x_4-x_2}{3} & \frac{x_4-x_3}{6} & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & \cdots & & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ \frac{y_3-y_2}{x_3-x_2} - \frac{y_2-y_1}{x_2-x_1} \\ \vdots \\ \frac{y_n-y_{n-1}}{x_n-x_{n-1}} - \frac{y_{n-1}-y_{n-2}}{x_{n-1}-x_{n-2}} \\ 0 \end{bmatrix}$$

To find the spline: solve $Sz = b$,

for $z = S^{-1}b$, where $z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$.

Next: Finite difference approximations (I.3)