09/24/2019

Lecture 6

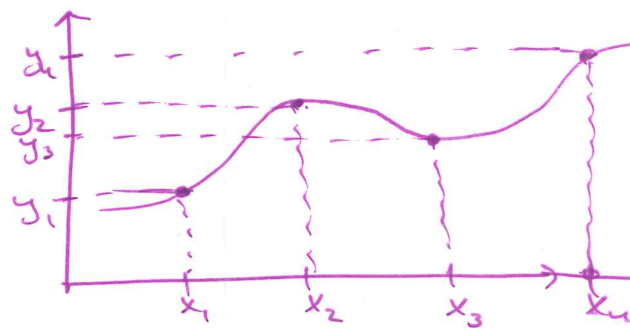
Announcements:

- Homework 2 is posted and is due on Tuesday, Oct 1 (1 week from now).
- Solutions to Quiz 1 are posted

Recap: Interpolation

Given $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$

Can we find a function $f(x)$ s.t. $f(x_1) = y_1, \dots, f(x_n) = y_n$?



I. Lagrange interpolation

II. Cubic splines interpolation

I. Lagrange interpolation.

Find $f(x) = a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$

a polynomial of degree $n-1$.

n unknowns: $a_1, a_2, \dots, a_n$

n equations: $f(x_1) = y_1, \dots, f(x_n) = y_n$

$$\begin{cases} a_1 x_1^{n-1} + a_2 x_1^{n-2} + \dots + a_n = y_1 \\ a_1 x_2^{n-1} + a_2 x_2^{n-2} + \dots + a_n = y_2 \\ \vdots \\ a_1 x_n^{n-1} + a_2 x_n^{n-2} + \dots + a_n = y_n \end{cases}$$

$$\begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & & & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

unknowns given

<u>Remarks</u>:

- A matrix of the form

$$\begin{bmatrix} x_1^{n-1} & \cdots & x_1 & 1 \\ \vdots & & & \\ x_n^{n-1} & \cdots & x_n & 1 \end{bmatrix}$$

is called a Vandermonde matrix.

- $$\det \begin{bmatrix} x_1^{n-1} & \cdots & x_1 & 1 \\ \vdots & & & \\ x_n^{n-1} & \cdots & x_n & 1 \end{bmatrix} = (-1)^{\frac{n(n-1)}{2}} \prod_{i>j} (x_i - x_j) \neq 0$$

whenever $x_i \neq x_j \ \forall i \neq j$.

$\Rightarrow$ System always has a unique solution

- However, it turns out that Vandermonde systems are often ill-conditioned.

$\boxed{\text{MATLAB}}$

II. Cubic spline interpolation

Given $(x_1,y_1), (x_2,y_2), \dots, (x_n,y_n)$

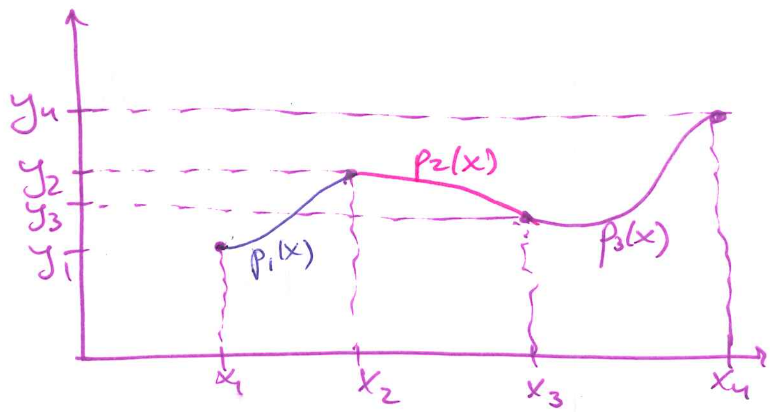
Fit a different polynomial

$p_j(x) = A_j x^3 + B_j x^2 + C_j x + D_j$

between x_j and x_{j+1}

$\forall j = 1, \dots, n-1.$

↳ cubic polynomial.



$4(n-1)$ unknowns $\{A_j, B_j, C_j, D_j : j = 1, \dots, n-1\}$.

~~In addition~~, Impose the following conditions:

(C1) Continuity and interpolation: $p_j(x_j) = y_j,\ p_j(x_{j+1}) = y_{j+1}$ $\forall j = 1, \dots, n-1$ $\boxed{2(n-1) \text{ eqs}}$

(C2). Differentiability: $p_j'(x_{j+1}) = p_{j+1}'(x_{j+1})$ $\forall j = 1, \dots, n-2$ $\boxed{(n-2) \text{ eqs}}$

(C3). Twice differentiable at $x_2, \dots, x_{n-1}$:

$p_j''(x_{j+1}) = p_{j+1}''(x_{j+1})$ $\forall j = 1, \dots, n-2.$ $\boxed{(n-2) \text{ eqs}}$

(C4). Impose $f''(x_1) = f''(x_n) = 0$ $\boxed{2 \text{ eqs}}$

$2(n-1) + (n-2) + (n-2) + 2 = 4(n-1)$ equations

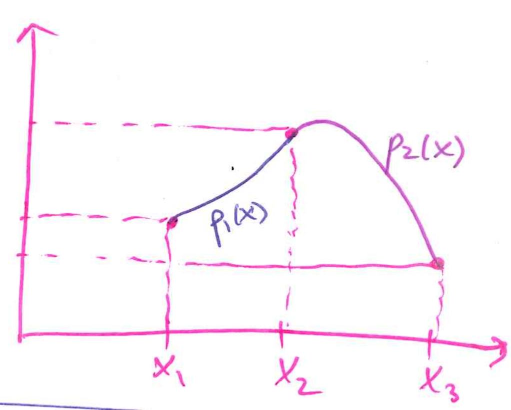
and $4(n-1)$ unknowns.

Example: Set $n = 3$. Let's solve.

$p_1(x) = A_1 x^3 + B_1 x^2 + C_1 x + D_1$

$p_2(x) = A_2 x^3 + B_2 x^2 + C_2 x + D_2.$

$\boxed{8 \text{ unknowns}}$



(C1) $p_1(x_1) = y_1$

$p_1(x_2) = y_2$

$p_2(x_2) = y_2$

$p_2(x_3) = y_3$

$$\boxed{\begin{aligned} &A_1 x_1^3 + B_1 x_1^2 + C_1 x_1 + D_1 = y_1 \\ &A_1 x_2^3 + B_1 x_2^2 + C_1 x_2 + D_1 = y_2 \\ &A_2 x_2^3 + B_2 x_2^2 + C_2 x_2 + D_2 = y_2 \\ &A_2 x_3^3 + B_2 x_3^2 + C_2 x_3 + D_2 = y_3 \end{aligned}}$$ 4 eqs.

(C2) $p_1'(x_2) = p_2'(x_2)$

$$\boxed{3A_1 x_2^2 + 2B_1 x_2 + C_1 = 3A_2 x_2^2 + 2B_2 x_2 + C_2}$$ 1 eq

rewrite $3A_1 x_2^2 + 2B_1 x_2 + C_1 - 3A_2 x_2^2 - 2B_2 x_2 - C_2 = 0.$

(C3) $p_1''(x_2) = p_2''(x_2)$

$$\boxed{6A_1 x_2 + 2B_1 = 6A_2 x_2 + 2B_2}$$ 1 eq.

rewrite $6A_1 x_2 + 2B_1 - 6A_2 x_2 - 2B_2 = 0$

(C4) $p_1''(x_1) = 0$

$p_2''(x_3) = 0$

$$\boxed{6A_1 x_1 + 2B_1 = 0}$$

$$\boxed{6A_2 x_3 + 2B_2 = 0}$$ 2 eq.

$\Rightarrow$ 8 equations
8 unknowns!

Let's write the linear system using matrix notation:

$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 \\ x_2^3 & x_2^2 & x_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2^3 & x_2^2 & x_2 & 1 \\ 0 & 0 & 0 & 0 & x_3^3 & x_3^2 & x_3 & 1 \\ 3x_2^2 & 2x_2 & 1 & 0 & -3x_2^2 & -2x_2 & -1 & 0 \\ 6x_2 & 2 & 0 & 0 & -6x_2 & -2 & 0 & 0 \\ 6x_1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6x_3 & 2 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$