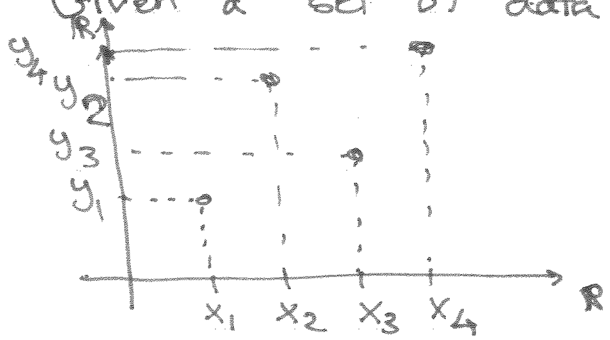
$$\Rightarrow \text{Cond}(A) = \frac{\max\{|\lambda_1|, \dots, |\lambda_n|\}}{\min\{|\lambda_1|, \dots, |\lambda_n|\}}$$

Ex: $A = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix} \Rightarrow \text{Cond}(A) = 1$

$$A = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & -2 & \\ 0 & & & 0.1 \end{bmatrix} \Rightarrow \text{Cond}(A) = \frac{|-2|}{|0.1|} = \frac{2}{0.1} = 20$$

I.2 Interpolation

Given a set of data pts $(x_1, y_1), \dots (x_n, y_n)$ in $\mathbb{R}^2$



Want: Fit a function $f(x)$ to the given data points. That is:
Find $f(x)$ such that
$f(x_1) = y_1, \dots, f(x_n) = y_n$

- Fill in missing data after making measurements at some sample points.

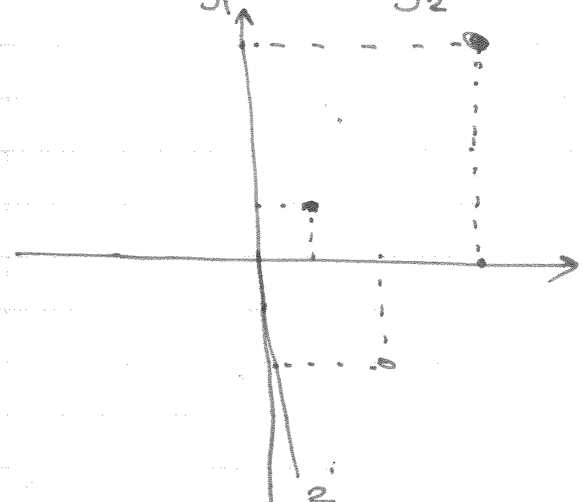
The interpolation problem is not well-posed: $\exists$ inf. many functions that we can fit into any finite data set.
Need to restrict the set of allowed functions.

Lagrange interpolation

Find a polynomial $p(x)$ of degree $n-1$ that fits this data.

$$p(x) = \underbrace{a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n}_{n \text{ unknowns}}$$

Ex: $x_1 = 1, \quad x_2 = 2, \quad x_3 = 4$
$y_1 = 1, \quad y_2 = -2, \quad y_3 = 7$ $\Big\}\ n = 3$



$$p(x) = a_1 x^2 + a_2 x + a_3$$

$$\begin{cases} p(1) = 1 \\ p(2) = -2 \\ p(4) = 7 \end{cases} \Rightarrow \begin{cases} a_1 + a_2 + a_3 = 1 \\ 4a_1 + 2a_2 + a_3 = -2 \\ 16a_1 + 4a_2 + a_3 = 7 \end{cases} \qquad \begin{array}{l} a_1(1^2) + a_2(1) + a_3 = 1 \\ a_1(4)^2 + a_2(2) + a_3 = -2 \\ a_1(4)^2 + a_2(4) + a_3 = 7 \end{array}$$

General case

$$\begin{cases} a_1 x_1^{n-1} + a_2 x_1^{n-2} + \cdots + a_{n-1} x_1 + a_n = y_1 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_1 x_n^{n-1} + a_2 x_n^{n-2} + \cdots + a_{n-1} x_1 + a_n = y_n \end{cases}$$

a linear system with n linear equations, n unknowns $(a_1, \dots, a_n)$

$$\underbrace{\begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix}}_{\text{known} \atop n \times n \text{ matrix}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\text{unknown}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\text{known}} \qquad (*)$$

Vandermonde matrix generated by $x_1, x_2, \dots, x_n$

Continuation of example

$$A = \begin{bmatrix} 1^2 & 1^1 & 1^0 \\ 2^2 & 2^1 & 2^0 \\ 4^2 & 4^1 & 4^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{bmatrix}$$

A in invertible, so $A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$ has a unique solution!

Fact:

$$\det \begin{bmatrix} x_1^{n-1} & \cdots & x_1 & 1 \\ x_2^{n-1} & \cdots & x_2 & 1 \\ \vdots & & & \\ x_n^{n-1} & \cdots & x_n & 1 \end{bmatrix} = (-1)^{\frac{n(n-1)}{2}} \prod_{i>j} (x_i - x_j)$$

For $n=3$ we get

$$\det \begin{bmatrix} x_1^2 & x_1 & 1 \\ \cdots & & \\ \cdots & & \end{bmatrix} = (-1)^{\frac{3\times 2}{2}} (x_2 - x_1)(x_3 - x_2)(x_3 - x_1)$$

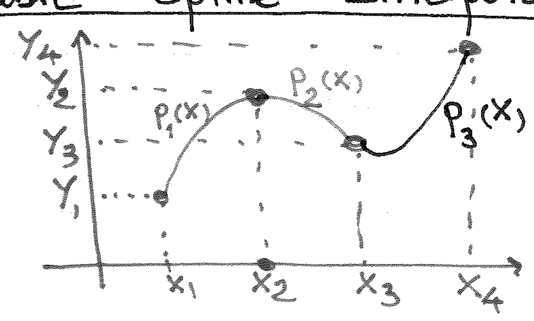
$\Rightarrow$ If $x_1, \ldots, x_n$ are all distinct, then the corresponding Vandermonde matrix is invertible.

But Vandermonde matrices are not in general well conditioned, specially for large n.

* If $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ solves the equation $A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

then $p(x) = a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ fits this data.

Cubic Spline Interpolation



• Given data (x_j, y_j), $j=1,\dots,n$. Fit the data with a piecewise-defined function

$$f(x) = \begin{cases} p_1(x) & ; \; x_1 \le x \le x_2 \\ p_2(x) & ; \; x_2 \le x \le x_3 \\ \vdots & \\ p_{n-1}(x) & ; \; x_{n-1} \le x \le x_n \end{cases}$$

there are $(n-1)$ pieces

such that

(a) $f(x)$ is continuous at all points (including $x_1, x_2, \dots, x_n$)

(b) more smoothness - possibly

- $f'(x)$ is well-defined,
- $f''(x)$ is well-defined, ...

Def'n: Functions like above where each p_j is a polynomial are called splines.

Our focus: We will require that

• $p_j(x)$ is a cubic functions (for each $j=1,\dots,n-1$), i.e.

$$\boxed{p_j(x) = A_j x^3 + B_j x^2 + C_j x + D_j}$$

unknown

$\Rightarrow$ $4(n-1)$ unknowns in total

So, to do cubic spline interpolation to n data points (x_j, y_j), we need to determine the value of $4(n-1)$ unknowns A_j, B_j, C_j, D_j.

Procedure: Impose the conditions of "smoothness":

(C1) $p_j(x_j) = y_j$ $\longrightarrow$ match the data

$p_j(x_{j+1}) = y_{j+1}$ $\longrightarrow$ continuity

both for $j = 1, 2, \dots, n-1$.

We get so far $2(n-1)$ equations

(C2) $p_j'(x_{j+1}) = p_{j+1}'(x_{j+1})$ for $j=1,\dots,n-2$

guarantees that f is differentiable everywhere (x_1, x_n)

$\hookrightarrow$ $n-2$ equations

(C3) $p_j''(x_{j+1}) = p_{j+1}''(x_{j+1})$, $j = 1, 2, \dots, n-2$
guarantees that f'' exists everywhere in (x_1, x_n)
↳ $n-2$ equations

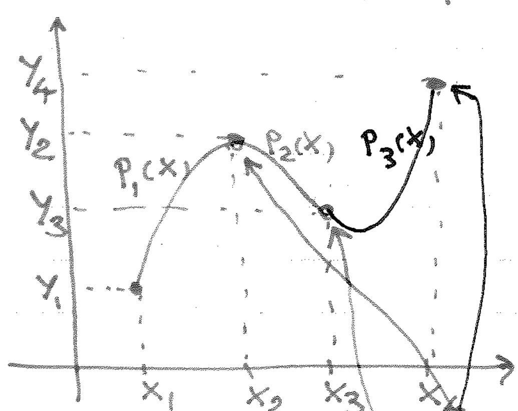
So:
there are $4(n-1) = 4n-4$ unknowns
there are $2(n-1) + 2(n-2) = 4n-6$ equations
We need two more conditions to get a unique solution.

(C4) We make $f''(x_1) = f''(x_2) = 0$

$$\Rightarrow \left.\begin{array}{l} p_1''(x_1) = 0 \\ p_{n-1}''(x_n) = 0 \end{array}\right\} \text{2 more equations}$$

$\Rightarrow$ $4n-4$ equations in total

Ex:

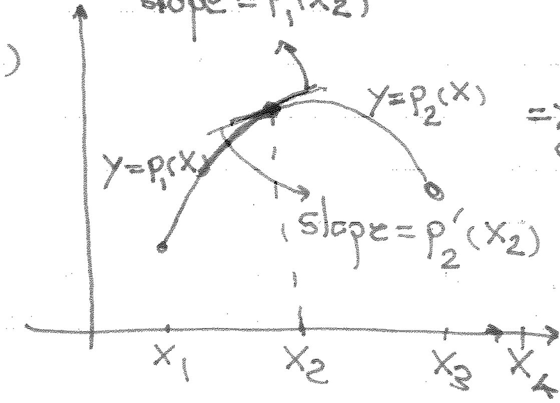


$p_1(x) = A_1x^3 + B_1x^2 + C_1x + D_1$
$p_2(x) = A_2x^3 + B_2x^2 + C_2x + D_2$
$p_3(x) = A_3x^3 + B_3x^2 + C_3x + D_3$

(C1) $p_1(x_1) = A_1x_1^3 + B_1x_1^2 + C_1x_1 + D_1 = y_1$
$p_2(x_2) = A_2x_2^3 + B_2x_2^2 + C_2x_2 + D_2 = y_2$
$p_3(x_3) = A_3x_3^3 + B_3x_3^2 + C_3x_3 + D_3 = y_3$
$p_4(x_4) = A_4x_4^3 + B_4x_4^2 + C_4x_4 + D_4 = y_4$

$p_1(x_2) = A_1x_2^3 + B_1x_2^2 + C_1x_2 + D_1 = y_2$
$p_2(x_3) = A_2x_3^3 + B_2x_3^2 + C_2x_3 + D_2 = y_3$
$p_3(x_4) = A_3x_4^3 + B_3x_4^2 + C_3x_4 + D_3 = y_4$

(C2)



$p_1'(x_2) = p_2'(x_2)$
$\Rightarrow 3A_1x_2^2 + 2B_1x_2 + C_1 = 3A_2x_2^2 + 2B_2x_2 + C_2$
Similarly
$p_2'(x_3) = p_3'(x_3)$
$p_3'(x_4) = p_4'(x_4)$

(C3) $p_1''(x_2) = p_2''(x_2)$

Since $p_j''(x) = 6A_j x + 2B_j$, we get

$\Rightarrow 6A_1 x_2 + 2B_1 = 6A_2 x_2 + 2B_2$

Similarly: $6A_2 x_3 + 2B_2 = 6A_3 x_3 + 2B_3$

$6A_3 x_4 + 2B_3 = 6A_4 x_4 + 2B_4$

(C4) $p_1''(x_1) = 0$

$\Rightarrow \boxed{6A_1 x_1 + 2B_1 = 0}$

$p_3''(x_4) = 0$

$\Rightarrow \boxed{6A_3 x_4 + 2B_2 = 0}$