Condition number

Want: solution of $Ax = b$ (A is an $n \times n$ matrix)

In many practices, matrices A are invertible but close to being singular.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.001 \end{bmatrix}$

$$\det(A) = 1.001 - 1 \times 1 = 0.001 \neq 0 \Rightarrow A \text{ invertible}$$

$$A^{-1} = \begin{bmatrix} 1001 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

. In any practical pbs, there will be errors:

(1) imperfect arithmetic in computers

(2) b is often obtained from some physical process (voltage, process, ...) and its value in uncertain.

$$Ax' = b + \Delta b$$

Assume A is invertible.

$$\Rightarrow x' = A^{-1}(b + \Delta b) =$$

$$= \underbrace{A^{-1} b}_{x} + \underbrace{A^{-1} \Delta b}_{\Delta x}$$

Want: Δx should be "small" when Δb is "small".

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.001 \end{bmatrix}$

① $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow x = A^{-1} b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$b' = \begin{bmatrix} 2 \\ 2.001 \end{bmatrix} = b + \underbrace{\begin{bmatrix} 0 \\ 0.001 \end{bmatrix}}_{\Delta b} \Rightarrow x' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Delta x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

<u>Let's relate Δb and Δx</u>

$$\Delta x = A^{-1}\Delta b$$

$$\Rightarrow \|\Delta x\| = \|A^{-1}\Delta b\|$$

$$\Rightarrow \|\Delta x\|.\|b\| = \|A^{-1}\Delta b\|\,\|b\| \qquad \|b\| \text{ 2-norm of } b$$

$$\Rightarrow \|\Delta x\|.\|b\| = \|A^{-1}\Delta b\|.\|Ax\|$$

(since $b = Ax$)

$$\|A^{-1}(\Delta b)\| \leqslant \|A^{-1}\|_{op}.\|\Delta b\|$$

$$\Rightarrow \|\Delta x\|.\|b\| \leqslant \|A^{-1}\|_{op}.\|\Delta b\|.\underbrace{\|Ax\|}_{\leqslant \|A\|_{op}\|x\|}$$

$$\Rightarrow \|\Delta x\|.\|b\| \leqslant \|A^{-1}\|_{op}.\|A\|_{op}.\|\Delta b\|.\|x\|$$

$$\Rightarrow \frac{\|\Delta x\|}{\|x\|} \leqslant \underbrace{\|A^{-1}\|_{op}.\|A\|_{op}}_{Cond(A)}\cdot\frac{\|Ab\|}{\|b\|}$$

$Cond(A) = \|A\|_{op}.\|A^{-1}\|_{op}$ "condition number of A"

$\shortparallel$

$K(A)$

We have

$$\frac{\|\Delta x\|}{\|x\|} \leqslant cond(A)\cdot\frac{\|\Delta b\|}{\|b\|}$$

Rmk: (1) $Cond(A) \geqslant 1$

Convention: $Con(A) = \infty$ if A is singular

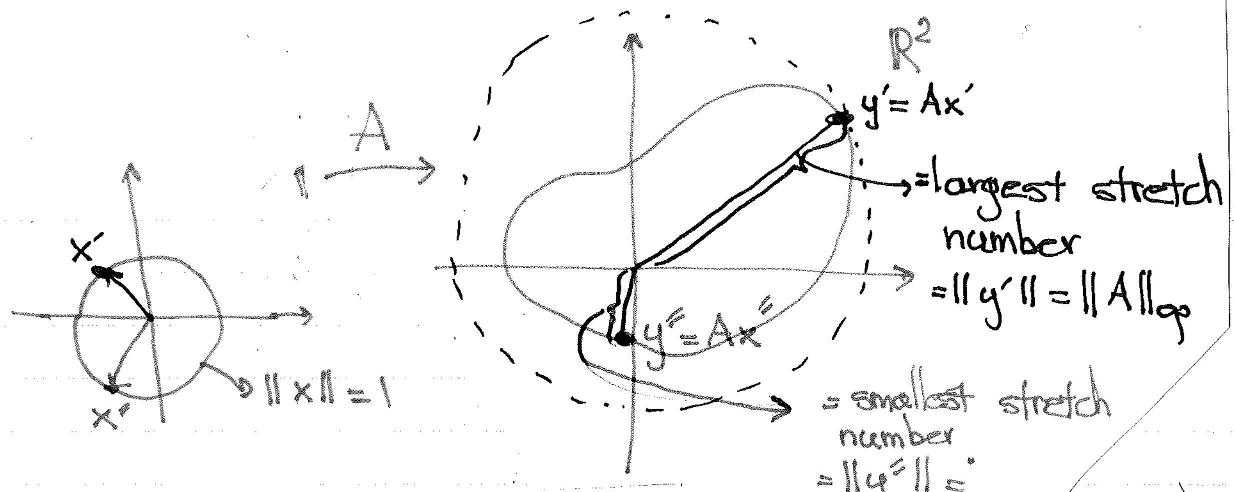
(2) We prefer $Cond(A)$ to be as close to 1 as possible

Ex: $A = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix}$

$Cond(A) = \|A\|_{op}.\|A^{-1}\|_{op} = 0.0001 \times 10000 = 1$

<u>Alternative interpretation</u>

$$\|A\|_{op} = \max_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \max_{\substack{\|x\|=1\\ x\neq 0}}\frac{\|Ax\|}{\underbrace{\|x\|}_{=1}} = \max_{\|x\|=1}\|Ax\|$$



$\|A\|_{op}$ largest stretch factor $\|A^{-1}\|_{op} = \frac{1}{\|y''\|}$

What about $\|A^{-1}\|_{op}$:

$$\|A^{-1}\|_{op} = \max_{y \neq \vec{0}} \frac{\|A^{-1}y\|}{\|y\|}$$

(change of variable $y = Ax$)
Can do this because for any $y \neq 0$, there is an x such that $y = Ax$

$$= \max_{x \neq \vec{0}} \frac{\|A^{-1}(Ax)\|}{\|Ax\|}$$

$$\|A^{-1}\|_{op} = \max_{x \neq \vec{0}} \frac{\|x\|}{\|Ax\|}$$

$$= \max_{x \neq \vec{0}} \frac{1}{\frac{\|Ax\|}{\|x\|}}$$

$$= \frac{1}{\min_{x \neq \vec{0}} \frac{\|Ax\|}{\|x\|}}$$ smallest stretch factor of A

$$\Rightarrow \text{Cond}(A) = \|A\|_{op} \cdot \|A^{-1}\|_{op}$$

$$= \frac{\text{largest stretch factor}}{\text{smallest stretch factor}}$$

<u>Let's compute this for a diagonal matrix</u>

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

A is invertible iff $\lambda_1, \lambda_2, \dots, \lambda_n \neq 0$

Then

$$A^{-1} = \begin{bmatrix} \frac{1}{\lambda_1^{-1}} & & & \\ & \frac{1}{\lambda_2^{-1}} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n^{-1}} \end{bmatrix}$$

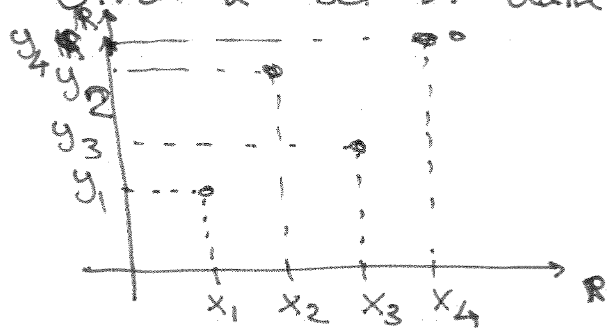
$$\Rightarrow \text{Cond}(A) = \frac{\max\{|\lambda_1|, \ldots, |\lambda_n|\}}{\min\{|\lambda_1|, \ldots, |\lambda_n|\}}$$

Ex: $A = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix} \Rightarrow \text{Cond}(A) = 1$

$A = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & -2 & \\ 0 & & & 0.1 \end{bmatrix} \Rightarrow \text{Cond}(A) = \frac{|-2|}{|0.1|} = \frac{2}{0.1} = 20$

I.2 Interpolation

Given a set of data pts $(x_1, y_1), \ldots (x_n, y_n)$ in $\mathbb{R}^2$



Want: Fit a function $f(x)$ to the given data points. That is;
Find $f(x)$ such that
$f(x_1) = y_1, \ldots, f(x_n) = y_n$

- Fill in missing data after making measurements at some sampl points.

The interpolation problem is not well-posed: $\exists$ inf. many function that we can fit into any finite data set.
Need to restrict the set of allowed functions.

Lagrange interpolation

Find a polynomial $p(x)$ of degree $n-1$ that fits this data.

$$p(x) = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

n unknowns