09/12/2019

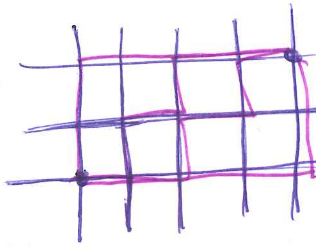
Lecture 3

Reminder: office hour today in AUDX 142

Recap: • Norms of vectors: ways to measure the magnitude / size of a vector.

(1). 2-norm $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ measures length.
Euclidean norm — Euclidean distance.

(2). 1-norm $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.
"Manhattan distance"



(3). ∞-norm $\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$.

(4). p-norm $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$. $1 \le p < \infty$

Fact: $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ for any $x \in \mathbb{R}^n$

Q: Find x s.t. $\|x\|_1 = \|x\|_2 = \|x\|_p = \|x\|_\infty$ for all p.

↳ $x = e_i$, $i = 1, 2, \ldots, n$

↳ $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i.$

Example: $x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$\|x\|_1 = 6$

$\|x\|_2 = \sqrt{1+4+9} = \sqrt{14} \approx 3.74$

$\|x\|_3 = (1^3 + |-2|^3 + 3^3)^{1/3} = (36)^{1/3} \approx 3.30$

$\|x\|_{30} = (1 + 2^{30} + \underbrace{3^{30}}_{\text{dominates}})^{1/30} \approx 3.0000005$

$\downarrow$

$\|x\|_\infty = 3.$

Formal definition of a norm:

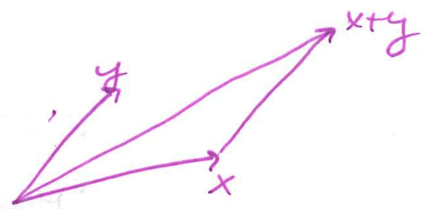
$\|x\|$ is a norm iff

(1). $\|x\| \geq 0$ with $\|x\| = 0 \Leftrightarrow x = 0$.

(2). $\forall$ vector $x \in \mathbb{R}^n$ and scalar $a \in \mathbb{R}$, $\|ax\| = |a| \, \|x\|$.

(3). $\forall x, y \in \mathbb{R}^n$ the triangle inequality holds:

$$\|x+y\| \leq \|x\| + \|y\|$$



Example 1: define $|||x||| = x_1 + x_2 + \cdots + x_n$

Q: Is this a norm on $\mathbb{R}^n$?

↳ (1) does not hold! if $x = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$, then $|||x||| = -n < 0$!!

Exercise: $|||\cdot|||$ violates properties (2) and (3) as well. Show!

Example 2: define $|||x||| = \min\{|x_1|, |x_2|, \ldots, |x_n|\}$.

Q: Is this a norm on $\mathbb{R}^n$?

(1) If $x = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, then $|||x||| = 0$, but $x \neq 0$!

(3). The triangle inequality doesn't hold:

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x+y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|||x+y||| = 1, \quad |||x||| = |||y||| = 0$$

and $1 > 0 + 0$

Example 3: How about $\|x\|_{0.5} = \left(|x_1|^{0.5} + |x_2|^{0.5} + \cdots + |x_n|^{0.5}\right)^2$

(i.e. set $p = 0.5$ in p-norm)

It can be shown that

- $\|x\|_p$ with $p\in(0,1)$ satisfies property (1)
- ~~XXXX~~ it also satisfies property (2). (show this!)
- However, $\|\cdot\|_p$ for $p\in(0,1)$ does NOT satisfy property (3), the triangle inequality.

$\Rightarrow$ $\|x\|_p$ is not a norm

Exercise: find a counter example.

However, $\|x+y\|_{0.5} \le 2\left(\|x\|_{0.5} + \|y\|_{0.5}\right)$

In fact, for any $p\in(0,1)$
$$\|x+y\|_p \le 2^{1/p - 1}\left(\|x\|_p + \|y\|_p\right)$$

We say that $\|\cdot\|_p$ with $p\in(0,1)$ is a quasi-norm.

I.1.6. Norms of matrices

There are many ways to define (useful) norms of matrices. Here are some important ones:

(1). Hilbert-Schmidt (Frobenius) norm (also 2-norm)

Think about A merely as a vector

Let $A = [a_{ij}]_{m\times n} := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$$\|A\|_{HS} = \|A\|_F := \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$

(2). The matrix norm or operator norm

Think of A as a linear operator/linear transformation

$A: \; x \longmapsto Ax \qquad \mathbb{R}^n \longrightarrow \mathbb{R}^m$

How ~~large~~ long can Ax get? How much does A stretch vectors?

Note that ~~#~~ $\|Ax\|_2$ depends on x, e.g. if we multiply x by 100, $\|Ax\|_2$ gets multiplied by 100.

That's why we consider the size of $\|Ax\|_2$ compared to $\|x\|_2$, i.e. how large is $\frac{\|Ax\|_2}{\|x\|_2}$?

↳ depends on x, not defined for $x=0$.

$$\|A\|_{op} := \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2}$$ measures the max factor by which A can stretch x.

Remarks:

(1). Note that for $x \neq 0$, $\|x\|_2 \neq 0$

$$\frac{\|Ax\|_2}{\|x\|_2} = \left\|\frac{1}{\|x\|_2} Ax\right\|_2 = \left\|A \frac{x}{\|x\|_2}\right\|_2$$

unit vector z, $\|z\|_2 = 1$

$$\boxed{\|A\|_{op} = \max_{\|z\|_2 = 1} \|Az\|_2 .}$$ equivalent definition.

(2). For any $x \neq 0$, $\frac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_{op}$.

$$\Leftrightarrow \boxed{\|Ax\|_2 \leq \|A\|_{op} \cdot \|x\|_2}$$

What is the operator norm for a diagonal matrix

$$A = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & d_n \end{bmatrix}$$

Want: $\|A\|_{op} = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$

For any $x \in \mathbb{R}^n$,

$$Ax = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}$$

$$\Rightarrow \|Ax\|_2^2 = (d_1x_1)^2 + (d_2x_2)^2 + \cdots + (d_nx_n)^2$$

$$= |d_1|^2|x_1|^2 + |d_2|^2|x_2|^2 + \cdots + |d_n|^2|x_n|^2$$

Let $D = \max\{|d_1|, |d_2|, \ldots, |d_n|\}$.

Then, $\|Ax\|_2^2 \le D^2(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)$

So, $\|Ax\|_2^2 \le D^2 \|x\|_2^2$

$\|Ax\|_2 \le D\|x\|_2$ for all $x \neq 0$

$$\Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \le D$$

$$\Rightarrow \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \le D$$

$$\Leftrightarrow \boxed{\|A\|_{op} \le D} \quad (*)$$

On the other hand, suppose $D = |d_k|$ for some $1 \le k \le n$.

Set $x = e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th}$ place.

$\boxed{(*) \,\&\, (**) \text{ imply } \|A\|_{op} = D.}$

Then $Ax = \begin{bmatrix} 0 \\ \vdots \\ d_k \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \frac{\|Ax\|_2}{\|x\|_2} = \frac{|d_k|}{1} = |d_k| = D.$

$$\Rightarrow \boxed{\|A\|_{op} = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \ge D} \quad (**)$$