Announcements

- Final exam review session: Thursday, Dec 5, 10:30am - 12pm in MATH 100
- Please fill out the teaching evaluations!
- If you are interested in undergraduate summer research opportunities, now is the time to start looking for a faculty, and to apply for a USRA or a Work Learn International undergraduate fellowship!

Stay tuned for:

- Extra office hours next week
- The problem set (just for exercise)

The power method

Given an $n \times n$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$ such that

- $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$
- $\{v_1, \ldots, v_n\}$ is an eigenbasis
- $\|v_i\|_2 = 1$ for all $j = 1, \ldots, n$.

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Given $x_0 = c_1 v_1 + + c_2 v_2 \cdots + c_n v_n$, where $c_1 \neq 0$, we have that

$$Ax_0 = c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n$$

$$\cdots$$

$$A^k x_0 = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 \cdots + c_n \lambda_n^k v_n$$

$$= \lambda_1^k \left(c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k v_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^k v_n \right)$$

$$= \lambda_1^k \left(c_1 v_1 + \varepsilon_k \right), \text{where } \varepsilon_k \to 0 \text{ as } k \to \infty.$$

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Thus,

$$\frac{A^k x_0}{\|A^k x_0\|} = \frac{\lambda_1^k (c_1 v_1 + \varepsilon_k)}{|\lambda_1^k \| c_1 v_1 + \varepsilon_k \|} \to \pm v_1$$

as $k \to \infty$.

The power method

Algorithm:

<u>Input:</u> Matrix A; random $x_0 \neq 0$; max number of iterations N.

<u>Iterate:</u> for $k = 1 : N$

$$x_k = Ax_{k-1};$$
$$x_k = \frac{x_k}{\|x_k\|};$$

end

<u>Output:</u> $v_1 = x_N$, $\lambda_1 = \langle v_1, Av_1 \rangle$.

Can we find any other eigenvectors/eigenvalues?

Given $s \in \mathbb{R}$, how can we find the closest eigenvalues to s?

Proposition: Let $s \in \mathbb{R}$, and suppose that A is $n \times n$ with real eigenvalues, and corresponding eigenbasis $\{v_1, \ldots, v_n\}$. Then, one of the following holds:

(a) s is an eigenvalue of A, i.e. $A - sI$ is not invertible; OR

(b) the eigenvalues of $(A - sI)^{-1}$ are exactly $\frac{1}{\lambda_1 - s}, \ldots, \frac{1}{\lambda_n - s}$, and the corresponding eigenvectors are still $v_1, \ldots, v_n$.

Proof: on board.

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<u>Consequence:</u> the eigenvalues of $(A - \lambda I)^{-1}$ are $\frac{1}{\lambda_1 - s}, \ldots, \frac{1}{\lambda_n - s}$, therefore, the dominant eigenvalue is $\frac{1}{\lambda_j - s}$ if and only if

$$\frac{1}{|\lambda_j - s|} > \frac{1}{|\lambda_i - s|}, \quad \forall i \neq j$$

$$\Leftrightarrow |\lambda_j - s| < |\lambda_i - s|, \quad \forall i \neq j,$$

i.e. λ_j is closest to s!

The closest eigenvalue to $s \in \mathbb{R}$.

Given $s \in \mathbb{R}$, we can use the power method on $(A - sI)^{-1}$ to get the eigenvector v_j that corresponds to the λ_j closest to s (if such a λ_j is unique). To get λ_j, just compute $\langle v_j, Av_j \rangle$.

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power iterations: $x_{k+1} = (A - sI)^{-1} x_k$,
in MATLAB: $x = (A - s * eye(n)) \backslash x$.

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How do we find the smallest $|\lambda_j|$? Choose $s = 0$, and perform the power method on A^{-1}!

Recurrence relations

A recursive definition of a sequence of numbers $a_0, a_1, a_2, \ldots$:

$$a_n = a_{n-1} + 2; \qquad a_0 = 5.$$

The sequence is

$$5, 7, 9, \ldots.$$

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But we can also find a formula for this sequence so we can quickly evaluate a_n for any n:

$$a_n = a_0 + 2n = 5 + 2n.$$

The Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

Here

$$F_{n+1} = F_n + F_{n-1}; \qquad F_0 = 0, F_1 = 1.$$

How can we find a formula for F_n?

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How can we find a formula for F_n?
Rewrite relation as a matrix equation:

$$F_{n+1} = F_n + F_{n-1}$$
$$F_n = F_n.$$

Thus,

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}, \quad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let $v_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then,

$$v_n = Av_{n-1}, \quad v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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$$\ldots F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

A more general example

$$x_{n+1} = 3x_n + x_{n-1} + 2x_{n-2}, \quad x_0 = a, x_1 = b, x_2 = c.$$

In matrix form:

$$\begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{bmatrix}; \quad v_0 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}.$$

$$v_{n-1} = Av_{n-2}, \quad v_0 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}.$$

Then,

$$v_n = A^n v_0.$$

Diagonalize A to get a formula for v_n, and, thus, for x_n.

Special facts about stochastic matrices.

Let P be an $n \times n$ stochastic matrix, i.e. $P_{ij} \geq 0$, and $\sum_{i=1}^{n} P_{ij} = 1$ for all j. Then,

1. If $v = [v_1, \ldots, v_n]^T$ is a state vector, i.e. $\sum v_i = 1$ and $0 \leq v_i \leq 1$, then Pv is also a state vector! (Proof: exercise).
2. $\lambda = 1$ is an eigenvalue of P. (Proof: on board.)
3. All other eigenvalues of P satisfy $|\lambda_j| \leq 1$.
4. If v is an eigenvector with eigenvalue 1, i.e., $Pv = v$, then, v has all nonnegative (or nonpositive) entries.
5. Eigenvectors corresponding to $|\lambda_j| < 1$ have entries that add up to 0.
6. When can we guarantee that $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j \neq 1$?
   If P or P^k for some $k \in \mathbb{N}$ has all positive entries, then $\lambda_1 = 1, |\lambda_j| < 1$ for all $j \neq 1$.