ALGORITHM (POWER METHOD) Lecture 25

Input: A ; random $x_0 \neq \vec{0}$; N

N: max # of iterations.

Note: We need that in $x_0 = c_1 v_1 + \cdots + c_n v_n$, $c_1 \neq 0$ in order for $\frac{A^k x_0}{\|A^k x_0\|}$ to converge to v_1!

But, if x_0 is choosen at random, this will be true.

Iterate: for $k = 1:N$

$x_k = A x_{k-1}$;

$x_k = \frac{x_k}{\|x_k\|_2}$;

end

Output: $v_1 = x_N$

$\lambda_1 = \langle v_1, A v_1 \rangle$

A modification:

Given $s \in \mathbb{R}$, how can we find the closest eigenvalue to s?

Proposition: Let $s \in \mathbb{R}$, and suppose A is $n \times n$ with real eigenvalues (as above). Then, one of the following holds:

(a) s is an eigenvalue of A, i.e. $A - sI$ is not invertible,

OR (b) The eigenvalues of $(A - sI)^{-1}$ are exactly $\frac{1}{\lambda_j - s}$ and the corresp. eigenvectors are still $v_1, \ldots, v_n$.

Proof: $Av = \lambda v, \quad v \neq \vec{0}$

$Av - sIv = \lambda v - sv$

$(A - sI)v = (\lambda - s)v$. If $\lambda = s$, then (a) holds.

Otherwise $s \neq \lambda_1, \ldots, \lambda_n$, so $A - sI$ is invertible, and

$\frac{1}{\lambda - s} v = (A - sI)^{-1} v$

i.e. $\frac{1}{\lambda - s}$ and v are an eigenvalue-eigenvector pair for $(A - sI)^{-1}$.

Eigenvalues of $(A-sI)^{-1}$ are $\frac{1}{\lambda_1 - s}, \ldots, \frac{1}{\lambda_n - s}$.

And the dominant eigenvalue is $\frac{1}{\lambda_j - s}$ if and only if

$$|\lambda_j - s| < |\lambda_\ell - s| \quad \text{for all } \ell \neq j.$$

Thus: Using the power method on $(A-sI)^{-1}$ will return the eigenvector v_j that corresponds to the λ_j closest to s. To get this λ_j, just compute $\langle v_j, Av_j \rangle$.

Important point: when computing $(A-sI)^{-1}x_k$, we don't actually need to compute $(A-sI)^{-1}$, but just $(A-sI)^{-1}x_k$. Thus, we need a solution to

$$(A-sI)y = x_k.$$

In MATLAB: $(A - s * \text{eye}(n)) \backslash x_k$.

Smallest eigenvalue in absolute value: power method on A^{-1}.

Recurrence relations (IV.4)

Examples: $\boxed{a_n = a_{n-1} + 2 \; ; \quad a_0 = 5}$ → recursive definition of a_n.

5, 7, 9, ...

Formula: $a_n = a_0 + 2n$

$$\boxed{a_n = 5 + 2n}$$

A more involved example (Fibonacci sequence)

$0, 1, 1, 2, 3, 5, 8, 13$

$$\boxed{F_{n+1} = F_n + F_{n-1}\ ; \quad F_0 = 0,\ F_1 = 1.} \quad (*)$$

recursion whose solution is the Fibonacci sequence.

Goal: Find a formula for F_n.

How? Rewrite this relation as a matrix equation!

$$F_{n+1} = F_n + F_{n-1}$$
$$F_n = F_n$$

$$\Rightarrow \left.\begin{array}{l} \underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{V_n} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}}_{V_{n-1}} \\ V_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\right\} (**)$$

$(*)$ and $(**)$ are equivalent.

Then, $V_1 = AV_0$; $V_2 = A^2 V_0$; $\dots$; $V_n = A^n V_0$.

That is: $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Noting that A is real symmetric (thus Hermitian), we know that A is unitarily decomposable.

Eigenvalues of A: $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Corresp. eigenvectors: $V_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$.

Then: $A = SDS^{-1}$ with $S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

Then, $A^n = SD^nS^{-1}$

$$\Rightarrow \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = SD^nS^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

$$= \cdots = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix}$$

$$\Rightarrow F_n = \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n).$$

A more-general example: $x_{n+1} = 3x_n + x_{n-1} + 2x_{n-2}$

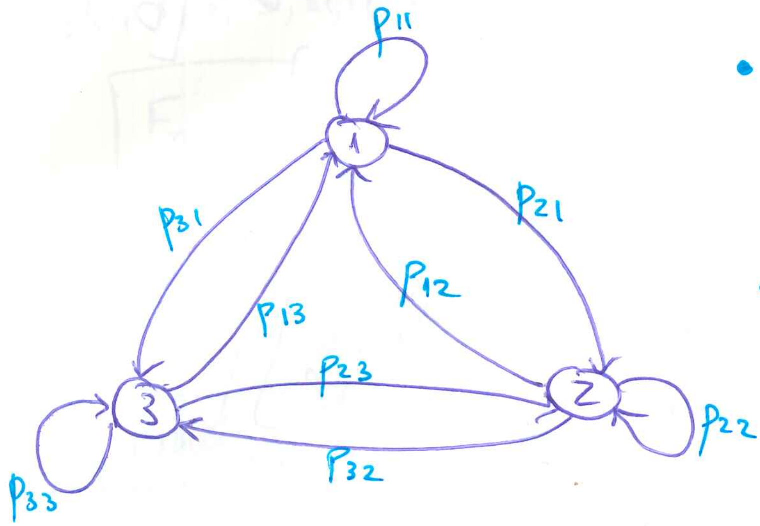
$x_0 = a;\ x_1 = b;\ x_2 = c.$

$$\underbrace{\begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix}}_{v_{n-1}} = \underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \end{bmatrix}}_{v_{n-2}} ; \quad v_0 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}. \qquad \begin{bmatrix} x_{n+2} \\ x_{n+1} \\ x_n \end{bmatrix}.$$

Then: diagonalize to get a formula for $v_n = A^n v_0$

and, thus, for x_n.

Markov chains



- p_{ij} : "**transition probability**" of getting from state j to state i.
- $0 \le p_{ij} \le 1$.
- $p_{1j} + p_{2j} + p_{3j} = 1$ for $j = 1, 2, 3$

getting from j to 1 — from j to 2 — from j to 3.

Let $x_{n,i}$ = probability of being at state i at time n.

Assume we make n transitions.

(e.g. Every day a bird is flying between islands (1), (2), and (3).
$x_{n,i}$ is the prob it is on island (i) on day n).

$x_{0,i}$ = prob of starting at state i

$x_{1,i}$ = prob of being at i after 1 step

$\vdots$

$x_{n,i}$ = . . .

Let $x_n = \begin{bmatrix} x_{n,1} \\ x_{n,2} \\ x_{n,3} \end{bmatrix}$ "distribution" after n steps.

Such vectors are called **state vectors**.

Note that
- $0 \le x_{n,i} \le 1$
- $\sum_{i=1}^{3} x_{n,i} = 1$.

Question: Given x_n, how do we find x_{n+1}?

$$x_{n+1,i} = x_{n,1}\, p_{i1} + x_{n,2}\, p_{i2} + x_{n,3}\, p_{i,3}$$

OR

$$x_{n+1} = \underbrace{\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}}_{P} x_n$$

Then, $\boxed{x_{n+1} = P x_n}$

$$= P^2 x_{n-1} = \cdots = P^{n+1} x_0.$$

So, $\boxed{x_n = P^n x_0}$ a matrix recursion

initial state.

To investigate the limit as $n \to \infty$ ("steady state" or "equilibrium", ...) if such a limit exists, we resort to eigenbasis as before, except that P has nice properties that will help!

Properties: $P = [p_{ij}]_{k \times k} = \begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & & \vdots \\ p_{k1} & \cdots & p_{kk} \end{bmatrix}$

- $0 \le p_{ij} \le 1$
- $\sum_{i=1}^{k} p_{ij} = 1 \quad \forall j.$

Such matrices are called stochastic.

Suppose P is stochastic and $\lim_{n\to\infty} P^n x_0$ exists and equals x.

Then, $P^N x_0 \approx x$ when N is large.

$$\left.\begin{array}{l} P^N x_0 \approx x \\[2ex] P^{N+1} x_0 \approx x \end{array}\right\} \Rightarrow x \approx P^{N+1} x_0 = P(P^N x_0) \approx Px$$

That is, $Px = x$.

i.e., If $\lim_{n \to \infty} P^n x_0 = x$, then x is a <u>stationary point</u>,

i.e., $Px = x$.

and $\lambda = 1$ is an eigenvalue of P with corresp. eigenvector x.

<u>Some special facts about stochastic matrices</u>

① If $v = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$ is a ~~state~~ vector (i.e. $\sum v_i = 1$, $1 \geq v_i \geq 0$)

then so is Pv (proof: exercise)

② P has an eigenvalue $\lambda = 1$:

<u>Proof</u>: $P^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \lambda = 1$ is an eigenval. of P^T

Since eigenvals of any matrix are same as those of its transpose,

$\Rightarrow \lambda = 1$ eigenvalue of P.

③. All other eigenvals of P satisfy $|\lambda_j| \leq 1$.

④. The eigenvector v_1 (for $\lambda = 1$) has non-negative (or non-positive) entries!

⑤. Eigenvectors corresp to $|\lambda_j| < 1$ have entries that add up to 0.

$[1 \cdots 1] P v_j = [1 \cdots 1] v_j = \sum v_i$, but $[1 \cdots 1] P v = [1 \cdots 1] \lambda v = \lambda \sum v_i$

$\Rightarrow \sum v_i = 0$

⑥ When can we guarantee that $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j \neq 1$?

If P or P^k for $k \in \mathbb{N}$ has all positive entries, then $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j \neq 1$.

(no 0 entries!)

Examples: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ stochastic, $P^2 = I$

$\lambda_1 = 1$, $\lambda_2 = -1$

$\lambda_1 = 1$ is <u>not</u> the dom. eig.

$$P = \begin{bmatrix} 3/4 & 0 & 0 \\ 1/8 & 3/4 & 1/4 \\ 1/8 & 1/4 & 3/4 \end{bmatrix}$$ stochastic.

not guaranteed to have $\lambda_1 = 1$ as dom. eig.

But using MATLAB:

$\lambda_1 = 1$; $v_1 = [0, 0.7071, 0.7071]^T$

$\lambda_2 = 0.75$; $v_2 = [0.8165, -0.4082, -0.4082]^T$

$\lambda_3 = 0.5$; $v_3 = [0, 0.7071, -0.7071]^T$

$\Rightarrow \lim_{n \to \infty} P^n x_0 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$. x_0 is a "state vector".

In general, $\lim_{n \to \infty} P^n x_0 = \pm \dfrac{v_1}{\|v_1\|}$ → so that entries are all nonnegative.