11/21/2019 Lecture 23

$$A\underbrace{v}_{\text{eigenvector } v\in\mathbb{C}^n} = \underbrace{\lambda}_{\text{eigenvalue } \lambda\in\mathbb{C}.} v$$

Example: $A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}.$

How to find all eigenvalues and corresponding eigenvectors?

1. let $p(\lambda) = \det(A-\lambda I) = \dots = -(\lambda-2)(\lambda-4)(\lambda-6)$

$\lambda_1 = 2,\ \lambda_2 = 4,\ \lambda_3 = 6$

2. $\lambda_1 = 2$: $(A-2I)v = \vec{0}$

$\Leftrightarrow \begin{bmatrix} 1 & -6 & -7 \\ 1 & 6 & 5 \\ -1 & -2 & 1 \end{bmatrix} v = \vec{0} \Leftrightarrow \dots\ v \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\} = E_2$

eigenspace for $\lambda = 2$.

$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda_1 = 2$.

$\lambda_2 = 4$: $(A-4I)v = \vec{0}$

$\begin{bmatrix} -1 & -6 & -7 \\ 1 & 4 & 5 \\ -1 & -2 & -3 \end{bmatrix} v = \vec{0} \Leftrightarrow \dots\ v \in \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\} = E_4$

eigenspace for $\lambda = 4$

$v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_3 = 6$: $\dots\ v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$

$E_6 = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}.$

Note: In this example we started with a 3×3 matrix A. and obtained 3 distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors v_1, v_2, v_3 that are l.i.

We say that $\{v_1, v_2, v_3\}$ is an <u>eigenbasis</u>.

<u>Theorem:</u> Let A be $n \times n$. If all n eigenvalues are distinct, then the corresponding eigenvectors ~~are~~ form a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$).

<u>Other possibilities</u>: <u>repeated eigenvalues</u>

<u>Example:</u> A could be such that $p(\lambda) = (\lambda-2)^2(\lambda-3)$

$$\Rightarrow \lambda_1 = \lambda_2 = 2 \;;\; \lambda_3 = 3.$$

* later OR $\quad p(\lambda) = (\lambda-1)^2(\lambda^2+1) = (\lambda-1)^2(\lambda-i)(\lambda+i).$

$$\Rightarrow \lambda_1 = \lambda_2 = 1; \; \lambda_3 = i, \; \lambda_4 = -i.$$

(complex eigenvalue is ok).

<u>Example:</u> Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$p(\lambda) = \det(A - \lambda I) = \det \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

$\lambda_1 = i, \quad \lambda_2 = -i.$

<u>$\lambda_1 = i$:</u> $(A - iI)v = \vec{0}$

$$\Leftrightarrow \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\lambda_1 = i, \; v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}}.$$

$v \in \text{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$

<u>$\lambda_2 = -i$:</u> $(A + iI)v = \vec{0}$

$$\Leftrightarrow \begin{bmatrix} i & -1 \\ 1 & -i \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$v \in \text{span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$

$$\boxed{\lambda_2 = -i, \; v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}}.$$

Useful fact: If $A \in \mathbb{R}^{n\times n}$ (i.e., its entries are real), and λ_1 is an eigenvalue of A with eigenvector v_1, then $\bar{\lambda}_1$ is also an eigenvalue of A with corresp. eigenvector $\bar{v}_1$.

Proof: $A v_1 = \lambda_1 v_1$

$\Leftrightarrow \quad \bar{A}\bar{v}_1 = \bar{\lambda}_1 \bar{v}_1$

$\Leftrightarrow \quad A\bar{v}_1 = \bar{\lambda}_1 \bar{v}_1$

as long as A is real-valued.

Repeated eigenvalues:

* Say $\det(A-\lambda I) = (\lambda-\lambda_1)^2(\lambda-\lambda_2)(\lambda-\lambda_3)^3$.

Then, the algebraic multiplicities of $\lambda_1, \lambda_2, \lambda_3$ are

$$m_1 = 2; \quad m_2 = 1; \quad m_3 = 3.$$

Next set $\dim(E_{\lambda_j}) = d_j$

d_j: geometric multiplicity of λ_j.

In our example from last lecture, we had A: 3×3 with $\lambda_1 = 2$; $\lambda_2 = 4$; $\lambda_3 = 6$ with $m_1 = m_2 = m_3 = 1$, $d_1 = d_2 = d_3 = 1$.

In general, $\boxed{1 \le d_j \le m_j}$.

Example: (1). $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $p(\lambda) = (\lambda-1)^2 \Rightarrow \lambda_1 = \lambda_2 = 1$.

$\boxed{m_1 = 2}$.

$A - \lambda_1 I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} v = \vec{0} \Rightarrow v \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

$E_1 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \Rightarrow \boxed{d_1 = 1}$.

(2) $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\Rightarrow p(\lambda) = (\lambda+1)^2 \Rightarrow \lambda_1 = -1, \boxed{m_1 = 2}$

$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \boxed{d_1 = 2}$.

a basis of $\mathbb{C}^n$ ~~of~~ $\{x_1, \dots, x_n\}$ s.t. ~~each~~ each x_j is an eigenvector of A

Theorem: There exists an eigenbasis corresp. to a matrix A iff $d_j = m_j \ \forall j$.

Diagonalization IV.1.9.

Setting: A $n \times n$

$\lambda_1, \lambda_2, \dots, \lambda_n$ eigenvalues. (s.t. $A v_j = \lambda_j v_j$).

$\{v_1, v_2, \dots, v_n\}$: eigenbasis.

Let: $S = [v_1 | v_2 | \cdots | v_n]$

(eigenvector v_j is in the j^{th} column of S)

Note: • S is $n \times n$.

• $AS = A[v_1 | \cdots | v_n] = [Av_1 | Av_2 | \cdots | Av_n]$

$= [\lambda_1 v_1 | \lambda_2 v_2 | \cdots | \lambda_n v_n]$

$= \underbrace{[v_1 | \cdots | v_n]}_{= S} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{:= D \text{ diagonal matrix!}}$

$\Rightarrow \boxed{AS = SD}$

Since S is invertible ($\{v_j\}$ is a basis), we have:

$\boxed{A = SDS^{-1}}$ → diagonalization of A.

Note also that $D = S^{-1}AS$.

Example: diagonalize $A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}$.

Solution: We know: $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 6$

$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

So: $A = SDS^{-1}$ with

$$S = \begin{bmatrix} 1 & -1 & -2 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Definition: A matrix A is diagonalizable if such S and D exist.

Why is diagonalization useful?

Suppose A is $n \times n$, $A = SDS^{-1}$.

(1). Determinant and trace. (2) Power.

Recall that • $\det(BC) = \det(B)\det(C)$

• $\det(B^{-1}) = 1/\det(B)$.

Then, $\det(A) = \det(S)\det(D)\det(S^{-1}) = \det(S)\det(D)\frac{1}{\det(S)}$

$\boxed{\det(A) = \det(D) = \lambda_1 \cdots \lambda_n}$ product of its eigenvalues (with multiplicities).

Trace: For $n \times n$ B, $\mathrm{trace}(B) := \sum_{i=1}^{n} B_{ii} =: \mathrm{tr}(B)$.

Fact: $\mathrm{tr}(BC) = \mathrm{tr}(CB)$, $\forall B, C$ (for which multiplying BC and CB both make sense).