11/19/2019 Orthonormal bases Lecture 22.

A basis $\{q_1, q_2, \ldots, q_m\}$ of a vector subspace V (real or complex)

is an <u>orthonormal basis</u> for V if

(in addition to being a basis)

- $\langle q_i, q_j \rangle = 0$ if $i \neq j$
- $\langle q_i, q_i \rangle = 1$ for each i.

alternatively,

$$\langle q_i, q_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

<u>Example</u>: (1) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis (ONB) for $\mathbb{R}^2$. (and $\mathbb{C}^2$).

(2) $\{e_1, e_2, \ldots, e_n\}$, where e_j are standard basis vectors in $\mathbb{R}^n$, is an ONB for $\mathbb{R}^n$ and also for $\mathbb{C}^n$.

(3) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$ is an ONB for $\mathbb{R}^2$ (and $\mathbb{C}^2$).

(4). $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1.1 \\ 1 \end{bmatrix} \right\}$ is not an ONB!

Let $\{q_1, \ldots, q_n\}$ be an ONB. for V. Then,

For any $v \in V$, $\exists$ a unique set of scalars $c_1, \ldots, c_n$

s.t. $v = c_1 q_1 + \cdots + c_n q_n$, <u>and we can find them as follows</u>:

We want $v = \underbrace{\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{c}$ coefficient vector.

To find c, we need to solve $Qc = v$ ← linear system.

Let's use the fact that $\{q_1, \dots, q_n\}$ is an ONB.

Claim: $Q^*Q = I_n$ ($n \times n$ identity matrix).

Proof:

$$Q^*Q = \begin{bmatrix} \bar{q}_1^T \\ \bar{q}_2^T \\ \vdots \\ \bar{q}_n^T \end{bmatrix} [q_1 | q_2 | \cdots | q_n]$$

$$= \begin{bmatrix} \bar{q}_1^T q_1 & \bar{q}_1^T q_2 & \cdots & \bar{q}_1^T q_n \\ \vdots & & & \\ \bar{q}_n^T q_1 & \bar{q}_n^T q_2 & \cdots & \bar{q}_n^T q_n \end{bmatrix} = \begin{bmatrix} \langle q_1, q_1 \rangle & \cdots & \langle q_1, q_n \rangle \\ \vdots & & \\ \langle q_n, q_1 \rangle & \cdots & \langle q_n, q_n \rangle \end{bmatrix}$$

$$= I_n$$

Thus, if we want $Qc = v$, then necessarily,

$$\underbrace{Q^*Q}_{I} c = Q^* v$$

$$\Leftrightarrow \quad c = Q^* v = \begin{bmatrix} \bar{q}_1^T \\ \bar{q}_2^T \\ \vdots \\ \bar{q}_n^T \end{bmatrix} v = \begin{bmatrix} \langle q_1, v \rangle \\ \langle q_2, v \rangle \\ \vdots \\ \langle q_n, v \rangle \end{bmatrix}.$$

Thus, $\boxed{c_j = \langle q_j, v \rangle}$!

And $v = \langle q_1, v \rangle q_1 + \langle q_2, v \rangle q_2 + \dots + \langle q_n, v \rangle q_n$.

whenever $\{q_1, \dots, q_n\}$ is an ONB.

An alternative way of finding the coefficients c_j:

since $v = c_1 q_1 + \dots + c_n q_n$, then

$$\langle q_j, v\rangle = \langle q_j, c_1 q_1\rangle + \langle q_j, c_2 q_2\rangle + \dots + \langle q_j, c_n q_n\rangle$$
$$= c_1 \langle q_j, q_1\rangle + c_2 \langle q_j, q_2\rangle + \dots + c_n \langle q_j, q_n\rangle$$
$$= c_j \langle q_j, q_j\rangle = c_j \text{ !!!.}$$

Orthogonal and unitary matrices.

A square matrix Q whose columns form an ONB is

- orthogonal (if all entries are real)
- unitary (if entries are complex).

Basic properties

① If Q is unitary (or orthogonal), then
$$Q Q^* = I_n = Q^* Q.$$
Thus, $Q^{-1} = Q^*$ and $(Q^*)^{-1} = Q$.

In the orthogonal case, we can replace Q^* with Q^T.

Remark: This implies that Q^* is unitary iff Q is unitary, so the rows of a unitary matrix also form an ONB!

②. A matrix is unitary $\Leftrightarrow$ $\|Qv\|_2 = \|v\|_2$ $\forall v \in \mathbb{C}^n$ ($\mathbb{R}^n$).

Proof: $\Rightarrow$: Suppose Q is unitary.

Parseval's identity: Let $Q = [q_1 | \dots | q_n]$, where $\{q_1, \dots, q_n\}$ is an ONB for V.
Then, $\forall v \in V$, $\sum_{j=1}^{n} |\langle q_j, v\rangle|^2 = \|v\|_2^2$.

Proof of Parseval's identity:

Let $v = c_1 q_1 + \cdots + c_n q_n$, where $c_j = \langle q_j, v \rangle$.

Then, $\|v\|_2^2 = \langle v, v \rangle = \left\langle \sum_{j=1}^n c_j q_j, \sum_{\ell=1}^n c_\ell q_\ell \right\rangle$

$$= \sum_{j=1}^n \sum_{\ell=1}^n \bar{c_j} c_\ell \underbrace{\langle q_j, q_\ell \rangle}_{=0 \text{ if } j \neq \ell}$$

$$= \sum_{j=1}^n \underbrace{\bar{c_j} c_j}_{=|c_j|^2} \underbrace{\|q_j\|_2^2}_{=1} = \sum_{j=1}^n |c_j|^2$$

$$= \sum_{j=1}^n |\langle q_j, v \rangle|^2.$$

For proof of $\Leftarrow$, see the notes.

Chapter IV. Eigenvalues and eigenvectors

Definition: Let A be an $n \times n$ (square) matrix.

A scalar λ (in $\mathbb{R}$ or $\mathbb{C}$) and a non-zero vector v (in $\mathbb{R}^n$ or $\mathbb{C}^n$) are an eigenvalue and an eigenvector pair for A if

$$Av = \lambda v.$$

Remarks: ① $v \neq \vec{0}$, but $\lambda = 0$ is ok! (I.e. when $v \in N(A)$, then $Av = \vec{0} = 0v$).

② $Av = \lambda v \iff Av = \lambda I_n v$

$\iff Av - \lambda I_n v = \vec{0} \iff (A - \lambda I_n)v = \vec{0}$

$\iff v \in N(A - \lambda I_n)$.

Since $v \neq \vec{0}$, this means that $(A-\lambda I_n)$ is singular, i.e.

$$\det(A-\lambda I_n)=0$$

<u>Set:</u> $p(\lambda)=\det(A-\lambda I_n)$. Then $p(\lambda)$ is a polynomial of degree n.

We call $p(\lambda)$ the <u>characteristic polynomial</u> of A.

<u>Theorem</u>: The roots of $p(\lambda)$ are eigenvalues of A!

<u>Proof</u>: We just showed that if λ is an eigenvalue, then $p(\lambda)=0$. On the other hand,

If $p(\lambda)=0$, then $A-\lambda I_n$ is singular

i.e. $\exists$ a non-zero vector in $N(A-\lambda I_n)$

so $(A-\lambda I_n)v=\vec{0}$

$\Leftrightarrow Av=\lambda v$.

<u>How do we find the eigenvalues and eigenvectors of a matrix A?</u>

(1) Find $p(\lambda)=\det(A-\lambda I)$

(2) Find the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of $p(\lambda)$, possibly complex, possibly repeated. These are the eigenvalues of A.

(3). For a given eigenvalue λ_j, the corresponding eigenvector(s): any non-zero v s.t. $(A-\lambda_j I)v=\vec{0}$

$$\Leftrightarrow v \in \underbrace{N(A-\lambda_j I)}_{\text{This is a subspace called the eigenspace } E_{\lambda_j}}$$

The eigenvectors corresponding to λ_j are all vectors in E_{λ_j}

We usually represent E_{λ_j} by a basis.

Examples: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}.$$

Solution: $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -6 & -7 \\ 1 & 8-\lambda & 5 \\ -1 & -2 & 1-\lambda \end{bmatrix}$

$= \underset{\text{after algebraic manipulation}}{\cdots\cdots} = -(\lambda-2)(\lambda-4)(\lambda-6)$

$\Rightarrow \lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 6.$

Corresponding eigenvectors:

$\lambda_1 = 2$: $(A - 2I)v = \vec{0}$

$\Leftrightarrow \begin{bmatrix} 1 & -6 & -7 \\ 1 & 6 & 5 \\ -1 & -2 & 1 \end{bmatrix} v = \vec{0}$

$\Leftrightarrow v \in \left\{ s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} = E_2$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

$\lambda_2 = 4$: $(A - 4I)v = \vec{0}$

$\cdots \Rightarrow E_4 = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$

$\lambda_3 = 6$: $\cdots$ $v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$

Note: In this example, we started with a 3x3 matrix A, and obtained 3 distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors v_1, v_2, v_3 that are linearly independent. We say that $\{v_1, v_2, v_3\}$ is an eigenbasis.

Theorem: let A be $n \times n$. If all n eigenvalues are distinct, then the corresponding eigenvectors form a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$).