11/14/2019 Lecture 21

Complex numbers:

$z = a + ib \in \mathbb{C}$, $i = \sqrt{-1}$.

$|z| = \sqrt{a^2 + b^2}$ modulus $Re(z) = a$, $Im(z) = b$.

$\bar{z} = a - ib$ conjugate to z

Properties of modulus:

- $|z_1 \cdot z_2| = |z_1||z_2|$
- $|cz| = |c|\,|z|$ for $c \in \mathbb{R}$, $z \in \mathbb{C}$.
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (provided $z_2 \neq 0$).

Properties of conjugation:

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

Useful facts:

(1) $\bar{x} = x \iff x \in \mathbb{R}$

(2) $Re(z) = \frac{1}{2}(z + \bar{z})$

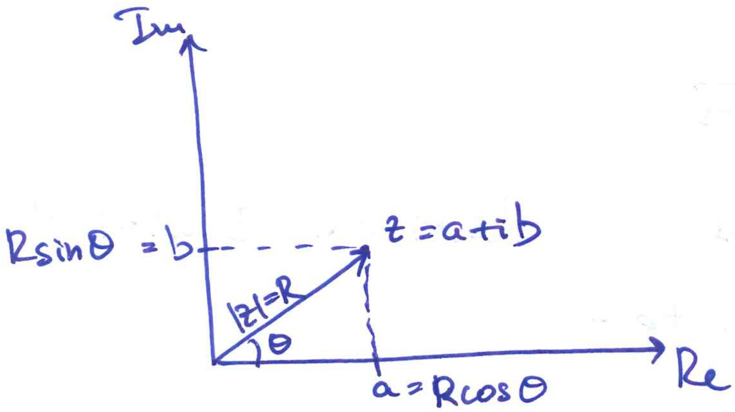
(3) $Im(z) = \frac{1}{2i}(z - \bar{z})$

Proof: If $z = a + ib$, then

$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(a + \cancel{ib} + a - \cancel{ib}) = \frac{2a}{2} = a = Re(z)$

$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2}(\cancel{a} + ib - \cancel{a} + ib) = \frac{2ib}{2i} = b = Im$

Geometric interpretation and polar form

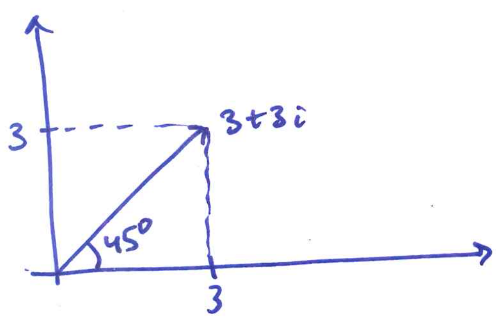


* $R = |z|$ is the modulus of z, $R = |z| = \sqrt{a^2+b^2}$
* θ is the angle z makes with the real axis.

Then, z can <u>uniquely</u> be represented by its "polar coordinates" R and θ:

Given R and θ, $z = R\cos\theta + iR\sin\theta = R(\cos\theta + i\sin\theta)$.

<u>Example</u>: Given $z = 3+3i$, find its polar coordinates.



$$\left.\begin{array}{l} |z| = \sqrt{18} = 3\sqrt{2} \\ \theta = 45^\circ = \frac{\pi}{4} \end{array}\right\} \text{Polar coordinates of } z: \left(3\sqrt{2}, \frac{\pi}{4}\right).$$

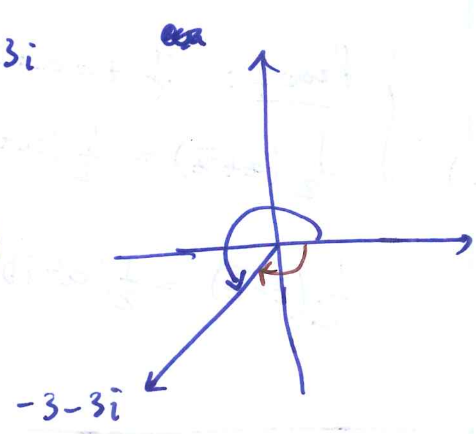
Thus, $z = 3+3i = 3\sqrt{2}\cos\frac{\pi}{4} + i\,3\sqrt{2}\sin\frac{\pi}{4}$

$= 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

<u>Example</u>: $z = -3-3i$

$|z| = 3\sqrt{2}$

$\theta = \frac{5\pi}{4}$ $\left(\text{or } -\frac{3\pi}{4}\right)$



__Remark__: Note that we can replace θ by $\theta + 2k\pi$, $k \in \mathbb{Z}$, and we still have

$$z = |z| \left(\underbrace{\cos(\theta + 2k\pi)}_{=\cos\theta} + i \underbrace{\sin(\theta + 2k\pi)}_{=\sin\theta} \right).$$

So, if we define

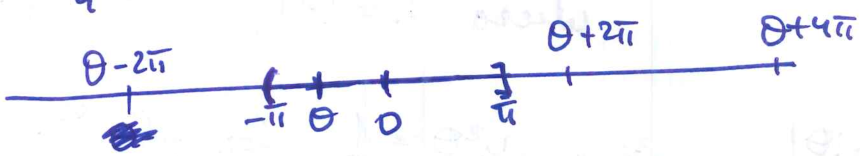
arg(z): the "angle" of z, then

$$\boxed{\underset{=\theta + 2k\pi}{\arg(z)} \text{ is not "single-valued"}}.$$

e.g., $\arg(3+3i) = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$

$\arg(-3-3i) = \frac{5\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$.

Let's restrict



__Define__:

$$\underset{\text{capital A}}{\mathrm{Arg}}(z) = \arg(z) \cap (-\pi, \pi]$$

__So__: $\mathrm{Arg}(3+3i) = \frac{\pi}{4}$ $\quad$ $\mathrm{Arg}(1) = 0$

$\mathrm{Arg}(-3-3i) = -\frac{3\pi}{4}$ $\quad$ $\mathrm{Arg}(i) = \frac{\pi}{2}$

$\mathrm{Arg}(-1) = \pi$, ... $\quad$ $\mathrm{Arg}(-i) = -\frac{\pi}{2}$

__Complex exponential__

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Euler's Formula

__Why is it defined this way?__

If $f(\theta) = \cos\theta + i\sin\theta$, then

$$f'(\theta) = -\sin\theta + i\cos\theta = i(\underbrace{i\sin\theta + \cos\theta}_{f(\theta)}) = i f(\theta)$$

Thus, $f(\theta) = \cos\theta + i\sin\theta$ solves the IVP: $\boxed{\begin{array}{l} f'(\theta) = i f(\theta) \\ f(0) = 1 \end{array}}$ Has a unique solution!

On the other hand, assume $e^{i\theta}$ is defined and satisfies the usual properties of the exponential function:

$$\left.\begin{array}{l} \frac{d}{d\theta} e^{i\theta} = i e^{i\theta} \\ \\ \text{and } e^{i0} = 1 \end{array}\right| \text{ So, } e^{i\theta} \text{ solves the IVP.}$$

So, we define $\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$

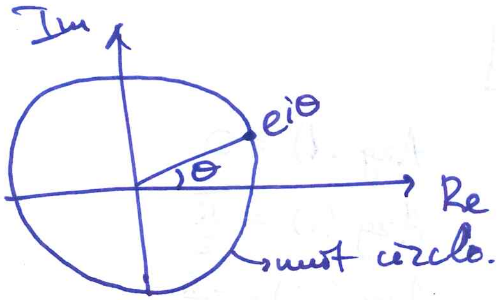
Thus, if $x = a+ib$, then $e^{x} = e^{a+ib} = e^{a} e^{ib} = e^{a}(\cos b + i\sin b)$.

Remarks:

(1). For every complex number z, $z = R(\cos\theta + i\sin\theta)$
$= Re^{i\theta}$!

where $R = |z|$, $\theta = Arg(z)$.

(2). $|e^{i\theta}| = \cos^2\theta + \sin^2\theta = 1$ for all θ.



(3). $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.

Proof: $e^{i\theta_1} e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$.

Thus, $(e^{i\theta})^n = e^{in\theta}$. $\gg$ It is easy to multiply complex numbers in polar form.

Example: $z_1 = 1+i$, $z_2 = i$, then calculate $z_1 \cdot z_2$; z_1^{40}; $z_1^3 z_2^3$.

Solution: $z_1 = 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$z_2 = i = 1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 1\cdot e^{i\frac{\pi}{2}}.$

Then, $z_1 z_2 = \sqrt{2}\cdot 1\, e^{i\left(\frac{\pi}{2}+\frac{\pi}{4}\right)} = \sqrt{2}\, e^{i\frac{3\pi}{4}} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= \sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) + i\sqrt{2}\cdot\frac{\sqrt{2}}{2}$

$= -1+i.$

$z_1^{40} = (\sqrt{2})^{40} e^{i\frac{\pi}{4}\cdot 40}$

$= 2^{20} e^{i\,10\pi}$

$= 2^{20} e^{i\cdot 0} = 2^{20}.$

$z_1^3 z_2^5 = (\sqrt{2})^3 e^{i\frac{3\pi}{4}} e^{i\frac{5\pi}{2}} = 2\sqrt{2}\, e^{i\left(\frac{3\pi}{4}+\frac{5\pi}{2}\right)} = 2\sqrt{2}\, e^{i\frac{13\pi}{4}} = 2\sqrt{2}\, e^{i\frac{5\pi}{4}} \ldots$

Complex vector spaces

- $\mathbb{C}^n = \left\{ z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_j \in \mathbb{C} \right\}.$
- scalars are $\mathbb{C}$
- All "basic" properties of $\mathbb{R}^n$ generalize to $\mathbb{C}^n$ after replacing real scalars with complex scalars (addition / subtraction / scalar multiplication)
- Need to change the definition of the inner product:

for $z, w \in \mathbb{C}^n$

$$\boxed{\langle w, z\rangle = \overline{w_1} z_1 + \overline{w_2} z_2 + \cdots + \overline{w_n} z_n}$$

Why do we need this change?

For $z \in \mathbb{C}^n$

$\|z\|_2 := \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$

want: $\langle z, z\rangle = \|z\|_2^2 = \overline{z_1} z_1 + \overline{z_2} z_2 + \cdots + \overline{z_n} z_n.$

Some consequences:

Let $s \in \mathbb{C}$, $z, w \in \mathbb{C}^n$. Then

(1) $\langle sw, z\rangle = \bar{s}\langle w, z\rangle$

$\langle w, sz\rangle = s\langle w, z\rangle$

(2). $\langle w, z\rangle = \overline{\langle z, w\rangle}$

(3). $\langle w, z\rangle = \overline{w}^T z$, where $\overline{w} = \begin{bmatrix} \overline{w}_1 \\ \vdots \\ \overline{w}_n \end{bmatrix}$.

(4). Recall that for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$

$$\langle Ax, y\rangle = \langle x, A^T y\rangle$$

If $z \in \mathbb{C}^n$, $w \in \mathbb{C}^m$, and $A \in \mathbb{C}^{m\times n}$, then

$$\langle Az, w\rangle = \langle z, \overline{A}^T w\rangle$$ (If $A = [a_{ij}]$, $\overline{A} = [\overline{a}_{ij}]$.

In fact, we <u>define</u> $\boxed{A^* := \overline{A}^T \text{ the } \underline{\text{adjoint}} \text{ of } A.}$

<u>Example</u>: $A = \begin{bmatrix} 1 & 1+2i & 3 \\ i & 2 & 1 \end{bmatrix}$

$$A^* = \begin{bmatrix} 1 & -i \\ 1-2i & 2 \\ 3 & 1 \end{bmatrix}.$$

MATLAB remarks: define i=sqrt(-1) just in case..

Then
- A' is actually A^*.
- A.' is A^T (no conjugates).