11/12/2019

Lecture 20

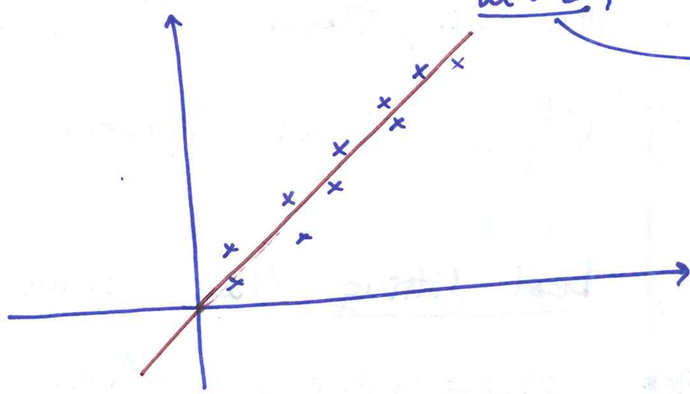
Polynomial Fit

Problem: Given n data points $(x_1, y_1), \dots, (x_n, y_n)$, find a polynomial approximation with a single polynomial of degree $m-1$: $\boxed{p(x) = a_1 x^{m-1} + a_2 x^{m-2} + \dots + a_{m-1} x + a_m}$

Note: If $m = n$, this is Lagrange interpolation.

Here, we are interested in $m < n$.

Example: $m = 2$, $n = 10$ → linear regression



How to solve the problem:

Given: $(x_1, y_1), \dots, (x_n, y_n)$

Want to fit: $p(x) = a_1 x^{m-1} + a_2 x^{m-2} + \dots + a_{m-1} x + a_m$

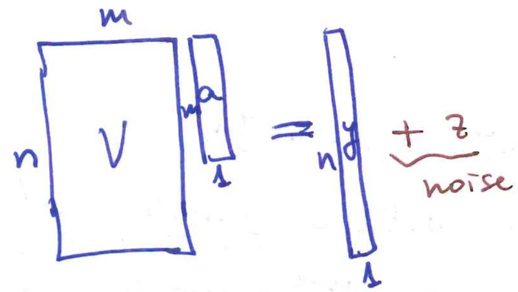
unknowns.

Want: $\boxed{p(x_1) = y_1, \dots, p(x_n) = y_n}$

(or at least approximately)

$$\underbrace{\begin{array}{c c} & m \\ n & \begin{bmatrix} x_1^{m-1} & x_1^{m-2} & \dots & x_1 & 1 \\ x_2^{m-1} & x_2^{m-2} & \dots & x_2 & 1 \\ \vdots & & & & \\ x_n^{m-1} & x_n^{m-2} & \dots & x_n & 1 \end{bmatrix} \end{array}}_{V_{n \times m} \text{ Vandermonde matrix}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m-1} \\ a_m \end{bmatrix}}_{a \text{ coefficients}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{y}$$

$$Va = y$$



This is the same as ~~[illegible]~~ last time.
We want to $\boxed{\text{project } y \text{ onto } \mathcal{R}(V)}$

- V is a tall matrix $(n > m)$. Thus, due to noise corruption, it is unlikely that $y \in \mathcal{R}(V)$.
- If $x_i \neq x_j$, then columns of V are linearly independent! (why?)
- Thus, want to solve $\boxed{V^T V a = V^T y}$!

Since columns of V are l.i., then $V^T V$ is invertible, and

$$a = (V^T V)^{-1} V^T y. \quad \text{least Squares solution}$$

Special case: $m = 2$, best fitting line (linear regression)

$$~~[illegible]~~ \quad p(x) = a_1 x + a_2 \leftarrow \text{line.}$$

Then,
$$V = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix}, \quad V^T V = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix}$$

$$V^T y = \begin{bmatrix} x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

$\boxed{\text{MATLAB}}$

So, to find a_1 and a_2, we need to solve:

$$\begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

Note: last time we looked at fitting

$$p(x_1, \dots, x_m) = b_1 x_1 + \dots + b_m x_m \boxed{+c}$$ "multiple linear regression"

and today: $p(x) = a_1 x + \underbrace{a_2}_{\text{constant term}}$

Complex vector spaces (III.2)

Review of complex numbers

Set $i = \sqrt{-1}$. Equivalently,

- $i^2 = -1$
- i is a root of $p(z) = z^2 + 1$. Thus, the solutions to $p(z) = 0$ are $z_1 = i$, $z_2 = -i$.
- $z^2 + 1 = (z+i)(z-i)$.

Define: $\mathbb{C} = \{ z = a + ib \,;\, a, b \in \mathbb{R} \}$.

Arithmetic in $\mathbb{C}$: let $z_1 = 2 + 3i$, $z_2 = 5 - 4i$

(a) $z_1 + z_2 = (2+5) + (3-4)i = 7 - i$.

(b) $z_1 - z_2 = (2-5) + i(3-(-4))i = -3 + 7i$.

(c) $3z_1 = 3.2 + 3.3i = 6 + 9i$

(d) $z_1 \cdot z_2 = (2+3i)(5-4i) = 10 + 15i - 8i - 12 i^2$, $i^2 = -1$

$= 10 + 12 + 7i = 22 + 7i$.

(e) division:

In general, let $z = a + ib$. Then,

(1). The <u>modulus</u> of z is

$$|z| = \sqrt{a^2+b^2}$$

e.g. $|3+4i| = \sqrt{9+16} = 5$

(2). <u>Complex conjugate</u> of z:

$$\bar{z} = a - ib$$

e.g. $\overline{(3+4i)} = 3-4i$

<u>Fact</u>: $z \cdot \bar{z} = |z|^2$ always real and non-negative

<u>Proof</u>: if $z = a+ib$, then $\bar{z} = a - ib$

$$\Rightarrow z \cdot \bar{z} = (a+ib)(a-ib) = a^2 + \cancel{iab} - \cancel{iab} - \underbrace{i^2}_{=-1} b^2$$

$$= a^2 + b^2 = |z|^2.$$

(3). <u>Real and imaginary parts</u> of z:

For $z = a + ib$,

$\operatorname{Re}(z) = a$ ← "real part" of z

$\operatorname{Im}(z) = b$ ← "imaginary part" of z.

$\operatorname{Re}(3+4i) = 3$

$\operatorname{Im}(3+4i) = 4$.

Back to

(e) division: If $z = a+ib$, then $z \cdot \bar{z} = |z|^2$

$$\Rightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{a-ib}{a^2+b^2}.$$

Thus, if

$z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $\boxed{\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2} = \frac{(a_1+ib_1)(a_2-ib_2)}{a_2^2+b_2^2}}$

Example: $z = \dfrac{3+4i}{5-12i} = \dfrac{(3+4i)(5+12i)}{25+144} = \dfrac{15+20i+36i+48i}{169}$

$$= \frac{-33+56i}{169} = -\frac{33}{169} + i\frac{56}{169}$$

Properties of modulus

- $|z_1 \cdot z_2| = |z_1||z_2|$

  $\Rightarrow |z_1^n| = |z_1|^n.$

Example: $|(1+i)^{20}| = |1+i|^{20} = (\sqrt{2})^{20} = 2^{10} = 1024.$

- $|cz| = |c| \cdot |z|$ for $c \in \mathbb{R}$, $z \in \mathbb{C}$.
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (provided $z_2 \neq 0$)

Properties of conjugation

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}.$

Useful facts:

(1) $x \in \mathbb{R} \Leftrightarrow \overline{x} = x$

(2) $\mathrm{Re}\{z\} = \frac{1}{2}(z + \overline{z})$

(3). $\mathrm{Im}\{z\} = \frac{1}{2i}(z - \overline{z})$