

Recap: • Linear systems of equations

$$Ax = b \quad \text{same as}$$

$\begin{matrix} \nearrow & \uparrow & \uparrow \\ m \times n & n \times 1 & m \times 1 \end{matrix}$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

• Any linear system will either have a unique solution, or infinitely many solutions, or no solution.

• If $m=n$ (i.e. n equations, n unknowns), we have

(1) $Ax = b$ has a unique solution iff $\det(A) \neq 0$

(2) ~~if~~ if $\det(A) = 0$, then system has either no solution or infinitely many solutions.

• How do we solve linear systems? $Ax = b$
 \hookrightarrow via Gaussian elimination.

Example:

$$x_1 + x_2 - 2x_3 = 3$$

$$x_1 - x_2 = -1$$

$$2x_1 - x_2 - x_3 = 0$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Augmented system:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 1 & -1 & 0 & -1 \\ 2 & -1 & -1 & 0 \end{array} \right] \begin{array}{l} r_2 = r_2 - r_1 \\ r_3 = r_3 - 2r_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & -2 & 2 & -4 \\ 0 & -3 & 3 & -6 \end{array} \right] \begin{array}{l} \\ r_2 = r_2 / (-2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & -3 & 3 & -6 \end{array} \right] \begin{array}{l} r_1 = r_1 - r_2 \\ r_3 = r_3 + 3r_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Reduced Row Echelon Form

pivots.

x_3 is a free variable.

x_1, x_2 are pivots.

$$\begin{cases} x_1 - x_3 = 1 \\ x_2 - x_3 = 2 \end{cases}$$

$$x_3 = t \text{ any real number}$$

$$x_1 = x_3 + 1 = t + 1$$

$$x_2 = x_3 + 2 = t + 2$$

$$\text{Solutions: } \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t+1 \\ t+2 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

Note: $\det \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix} = 1 \times \det \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} - 1 \times \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + (-2) \times \det \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$$= 1 + 1 + (-2) \times (-1 + 2) = 2 - 2 = 0$$

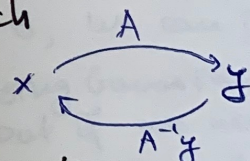
Indeed, the system has infinitely many solutions.

Now, suppose $Ax = y$, A is $n \times n$ and $\det(A) \neq 0$.

In this case, there is a unique solution^{*} for any y !

In fact $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a "function" which

maps $x \mapsto y = Ax$



and it is invertible, i.e. we can define A^{-1} as the map that sends y to x above, i.e.

$$Ax = y \quad (\Leftrightarrow) \quad A^{-1}y = x.$$

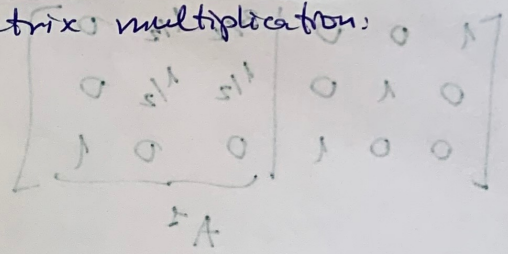
FACT: A^{-1} is an $n \times n$ matrix.

Next, define $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$ the $n \times n$ identity matrix.

It turns out that $I_n x = x \quad \forall x \in \mathbb{R}^n$.

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

I_n takes on the role of 1 in matrix multiplication.



- $A \cdot I_n = A \quad \forall n \times n \ A$
- $I_n \cdot B = B \quad \forall n \times n \ B$
- $I_n x = x \quad \forall x \in \mathbb{R}^n$

~~Combining~~ Combining, if A is $n \times n$ with $\det(A) \neq 0$,

then (1) A^{-1} exists and is an $n \times n$ matrix, ~~and~~ and A is invertible

(2) $(A^{-1})A = A(A^{-1}) = I_n$

(3) Given the system $Ax = b$, multiplying both sides with A^{-1} on the left, we get

$$\underbrace{A^{-1}A}_I x = A^{-1}b$$

$x = I_n x = A^{-1}b \Rightarrow$ the unique solution is $x = A^{-1}b$

So, how do we compute A^{-1} ?

\hookrightarrow using Gaussian elimination!

Build $\left[\begin{array}{c|c} A & I_n \end{array} \right]$ $\xrightarrow{\text{Gaussian elimination}}$ until we get I_n on the left.

$\longrightarrow \left[\begin{array}{c|c} I_n & A^{-1} \end{array} \right]$

Example: let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Build $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$

$\xrightarrow{r_2 = r_2 - r_1} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 = r_2/2} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 = r_1 + r_2}$

If we just had to solve 1 system $Ax = b$, we can just solve for x using Gaussian elimination, but if we wanted to solve many systems $Ax = b_i$, then, ~~we~~ it would make sense to compute A^{-1} .

$$\begin{bmatrix} 1 & 0 & 0 & | & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & | & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

A^{-1}

MATLAB script.

(1). How to multiply matrices and vectors.

$$\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 33 \\ -25 \end{bmatrix}$$

(2) Solve $\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Norms of Vectors and Matrices

Norms of vectors (I.1.5)

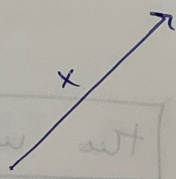
Given a vector $x \in \mathbb{R}^n$, how can we define its magnitude/size?

Example: $x \in \mathbb{R}$ ($n=1$), $|x|$ does this.

There are (infinitely) many ways to do this for vectors in \mathbb{R}^n .

Most commonly used norms: in \mathbb{R}^n :

(1) $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$



If $x = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$

$$\|x\|_2 = \sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5.$$

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Euclidean norm or 2-norm

This is the length ~~of the vector x~~ of the vector x .

(2) $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$\|x\|_1 := |x_1| + \dots + |x_n|$$

one-norm or l_1 -norm of x

If $x = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$,

$$\|x\|_1 = |3| + |4| + |0| = 7.$$

$$\begin{bmatrix} 0 & 0 & 1 & | & 0 & 1 & 1 \\ 0 & 1 & 0 & | & 0 & 1 & 1 \\ 1 & 0 & 0 & | & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = A$$

example: algebra

$$\begin{bmatrix} 0 & 0 & 1 & | & 0 & 1 & 1 \\ 0 & 1 & 0 & | & 0 & 1 & 1 \\ 1 & 0 & 0 & | & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & | & 0 & 1 & 1 \\ 0 & 1 & 0 & | & 0 & 1 & 1 \\ 1 & 0 & 0 & | & 1 & 0 & 0 \end{bmatrix}$$

$$17 - 57 = 57$$

Q: Is it possible to have

$\|x\|_2 > \|x\|_1$? No

$\|x\|_2 = \|x\|_1$? yes

If $x = \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\|x\|_2 = |a| = \|x\|_1$.

If turns out that $\|x\|_2 \leq \|x\|_1 \quad \forall x \in \mathbb{R}^n$.

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$$

$$\Leftrightarrow \|x\|_2^2 \leq \|x\|_1^2$$

$$\|x\|_2^2 = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 \stackrel{?}{\leq} (|x_1| + |x_2| + \dots + |x_n|)^2$$

$$\Leftrightarrow |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 \stackrel{?}{\leq} |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 + \sum_{i < j} 2|x_i||x_j|$$

$$\Leftrightarrow 0 \stackrel{?}{\leq} \sum_{i < j} 2|x_i||x_j| \quad \text{True}$$

Thus, $\|x\|_2 \leq \|x\|_1 \quad \forall x \in \mathbb{R}^n$.

(3). ∞ -norm ("infinity norm")

For $x = [x_1, \dots, x_n]^T$

$$\|x\|_\infty = \max \{ |x_1|, \dots, |x_n| \}$$

Example: $y = [2, -5, -3, 4, -7]^T$, $\|y\|_\infty = 7$.

(4). p -norm ($1 \leq p \leq \infty$)

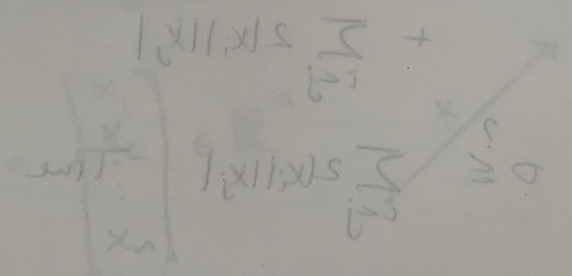
$$x = [x_1, x_2, \dots, x_n]^T, \quad \|x\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{1/p}$$

Note: $p=1 \rightarrow 1$ -norm
 $p=2 \rightarrow 2$ -norm

$\|x\|$ is a norm if

- (1) $\|x\| \geq 0$ $\forall x$ and $\|x\|=0 \Leftrightarrow x=0$
- (2) $\forall x \in \mathbb{R}^n, a \in \mathbb{R} \quad \|ax\| = |a|\|x\|$
- (3) triangle inequality: $\|x+y\| \leq \|x\| + \|y\|$

$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$
 $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$
 $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$



(3) ∞ -norm: $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$
 For $x = [x_1, \dots, x_n]^T$

Example: $x = [1, 2, 3, 4, 5]^T$
 $\|x\|_1 = 1+2+3+4+5 = 15$
 $\|x\|_2 = \sqrt{1^2+2^2+3^2+4^2+5^2} = \sqrt{55}$
 $\|x\|_\infty = 5$

Note: $\|x\|_p$ for $p \geq 1$ is a norm.
 For $p < 1$, it is not a norm.