09/10/2019 Lecture 2 MATH 307 Section 103

Recap: • Linear systems of equations

$$\underset{m\times n}{A}\;\underset{n\times 1}{x} = \underset{m\times 1}{b} \quad \text{same as} \quad \left| \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{array} \right.$$

• Any linear system will either have a <u>unique solution</u>, or <u>infinitely many solutions</u>, or <u>no solution</u>.

• If $m=n$ (i.e. n equations, n unknowns), we have

(1) $Ax=b$ has a unique solution iff $\det(A)\neq 0$

(2) [redacted] If $\det(A)=0$, then system has either no solution or infinitely many solutions.

• How do we solve linear systems? $Ax=b$

↳ via Gaussian elimination.

Example:

$$\begin{aligned} x_1+x_2-2x_3&=3 \\ x_1-x_2&=-1 \\ 2x_1-x_2-x_3&=0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Augmented system:

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 3 \\ \underline{1} & -1 & 0 & -1 \\ \underline{2} & -1 & -1 & 0 \end{array}\right] \xrightarrow[r3=r3-2r1]{r2=r2-r1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & \boxed{-2} & 2 & -4 \\ 0 & -3 & 3 & -6 \end{array}\right] \xrightarrow{r2=r2/(-2)}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & -3 & 3 & -6 \end{array}\right] \xrightarrow[r3=r3+3r2]{r1=r1-r2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row Echelon Form

pivots.

x_3 is a <u>free</u> variable.

x_1, x_2 are <u>pivots</u>.

$$\left| \begin{array}{l} x_1 - x_3 = 1 \\ x_2 - x_3 = 2 \end{array} \right.$$

$x_3 = t$ any real number

$x_1 = x_3 + 1 = t + 1$

$x_2 = x_3 + 2 = t + 2$

Solutions: $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 1+t \\ 2+t \end{bmatrix} : t \in \mathbb{R} \right\}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

Note: $\det \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix} = 1 \times \det \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} - 1 \times \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + (-2) \times \det \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$$= 1 + 1 + (-2) \times (-1 + 2) = 2 - 2 = 0$$

Indeed, the system has infinitely many solutions.

Now, suppose $Ax = y$, A is $n \times n$ and $\det(A) \neq 0$.

In this case, there is a unique solution x for any y!

In fact $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a "function" which maps

$$x \longmapsto y = Ax$$

$$\begin{tikzcd} x \arrow[r, bend left, "A"] & y \arrow[l, bend left, "A^{-1}y"] \end{tikzcd}$$

and it is invertible, i.e. we can define A^{-1} as the map that sends y to x above, i.e.

$$Ax = y \iff A^{-1}y = x.$$

FACT: A^{-1} is an $n \times n$ matrix.

Next, define $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 1 \end{bmatrix}$ the $n \times n$ identity matrix.

It turns out that $I_n x = x \quad \forall x \in \mathbb{R}^n$. $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

I_n takes on the role of 1 in matrix multiplication:

- $A \cdot I_n = A \quad \forall\ m \times n\ A$
- $I_n \cdot B = B \quad \forall\ n \times m\ B$
- $I_n x = x \quad \forall x \in \mathbb{R}^n$

Combining, if A is $n \times n$ with $\det(A) \neq 0$, then

(A) A^{-1} exists and is an $n \times n$ matrix, and A is <u>invertible</u>

(2) $(A^{-1})A = A(A^{-1}) = I_n$.

(3) Given the system $Ax = b$, multiplying both sides with A^{-1} on the left, we get

$$\underbrace{A^{-1}A}_{I_n}x = A^{-1}b$$

$$x = I_n x = A^{-1}b \Rightarrow \boxed{\text{the unique solution is } x = A^{-1}b}$$

So, how do we compute A^{-1}?

↳ using Gaussian elimination!

Build $\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{Gaussian elimination}}$ until we get I_n on the left.

$\cdots \longrightarrow \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$

If we just had to solve 1 system $Ax = b$, we can just solve for x using Gaussian elimination, but if we wanted to solve many systems $Ax = b_1$, $\vdots$ then, it would make sense to compute A^{-1}.

Example: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Build $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{r2 = r2 - r1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r2 = r2/2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r1 = r1 + r2}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

A^{-1}

MATLAB script.

(1). How to multiply matrices and vectors.

$$\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 33 \\ -29 \end{bmatrix}$$

(2) Solve $\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Norms of Vectors and Matrices

Norms of vectors (I.1.5)

Given a vector $x \in \mathbb{R}^n$, how can we define its magnitude/size?

Example: $x \in \mathbb{R}$ $(n=1)$, $|x|$ does this.

There are (infinitely) many ways to do this for vectors in $\mathbb{R}^n$:

Most commonly used norms on $\mathbb{R}^n$:

(1). $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

If $x = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$

$$\|x\|_2 = \sqrt{3^2+4^2+0^2} = \sqrt{25} = 5.$$

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Euclidean norm or 2-norm

This is the length of the vector x.

(2) $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ $\quad \|x\|_1 := |x_1| + \dots + |x_n|$ $\quad$ one-norm or l_1-norm of x

If $x = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\|x\|_1 = |3| + |4| + |0| = 7.$

Q: Is it possible to have

- $\|x\|_2 > \|x\|_1$? No
- $\|x\|_2 = \|x\|_1$? yes

  If $x = \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\|x\|_2 = |a| = \|x\|_1$.

It turns out that $\|x\|_2 \leq \|x\|_1 \quad \forall x \in \mathbb{R}^n$:

$$\Leftrightarrow \|x\|_2^2 \leq \|x\|_1^2$$

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

$$\|x\|_2^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \overset{?}{\leq} (|x_1| + |x_2| + \cdots + |x_n|)^2$$

$$\Leftrightarrow |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \overset{?}{\leq} |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 + \sum_{i<j} 2|x_i||x_j|$$

$$\Leftrightarrow \quad 0 \overset{?}{\leq} \sum_{i<j} 2|x_i||x_j| \quad \text{True}$$

Thus, $\|x\|_2 \leq \|x\|_1 \quad \forall x \in \mathbb{R}^n$.

(3). <u>∞-norm</u> ("infinity norm")

For $x = [x_1, \ldots, x_n]^T$

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

<u>Example</u>: $y = [2, -5, -3, 1, -7]^T$, $\|y\|_\infty = 7$.

(4). <u>p-norm</u> $(1 \leq p \leq \infty)$

$x = [x_1, x_2, \ldots, x_n]^T$, $\|x\|_p := \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$

<u>Note</u>: $p = 1 \rightsquigarrow$ 1-norm

$p = 2 \rightsquigarrow$ 2-norm

$\|x\|$ is a norm if

(1) $\|x\| \geq 0$ $\forall x$ and $\|x\| = 0 \iff x = 0$

(2) $\forall x \in \mathbb{R}^n$, $a \in \mathbb{R}$ $\|ax\| = |a| \|x\|$.

(3) triangle ineq. $\|x+y\| \leq \|x\| + \|y\|$