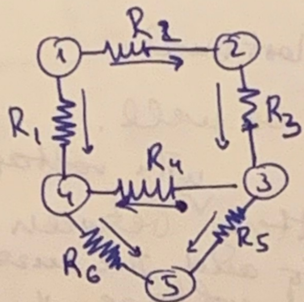
10/31/2019

Graphs & Networks through an example

1. Consider the following resistor network



(a). Construct the incidence matrix D.

(b). Construct the Laplacian L, for $R_i = 1$, $i = 1, \dots, 6$.

(c). Let $\vec{v}$ be the vector of voltages at each of the vertices.

$\vec{z}$ vector of currents at each of the vertices

$\vec{j}$ vector of currents at each of the edges.

~~(d).~~ Express $\vec{z}$ in terms of $\vec{j}$,

$\vec{j}$ in terms of $\vec{v}$, (Ohm's Law).

and $\vec{z}$ in terms of $\vec{v}$

(d). State Kirchoff's law and what it implies in terms of $\vec{z}$, $\vec{v}$, and $\vec{j}$.

<u>Solutions</u>: (a).

$$D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \end{array} ; \quad L = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} \begin{array}{c} \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \end{array}$$

(b). $L = D^T R^{-1} D$, $\quad L_{ii} = \sum_j \frac{1}{R_j}$ sum over all edges incident to i

$L_{ij} = -\frac{1}{R_k}$ $\exists$ $i \overset{R_k}{—} j$

(c). $\vec{z} = D^T \vec{j}$; $\underbrace{R\vec{j} = D\vec{v}}_{\text{Ohm's Law}} \Rightarrow \vec{j} = R^{-1} D \vec{v}$; $\vec{z} = D^T R^{-1} D \vec{v} = L\vec{v}$.

(d). Kirchoff's law: no current accumulates at any vertex unless there are power sources involved.

$$\Rightarrow \vec{z} = \vec{0}, \text{ i.e. } \mathbb{V}\vec{v} = \vec{0} \Leftrightarrow D\vec{v} = \vec{0}$$

from last time?

Since $\vec{j} = R^{-1}D\vec{v} \Rightarrow \vec{j} = \vec{0}$ as well.

(e) Suppose we add a battery with voltages $v_2 = 6$, $v_4 = 0$ between nodes ② and ④.
Find the voltages at v_1, v_3, v_5 and the currents v_2, v_4.
The terminals are kept at a fixed voltage $v_2 = b_2$, $v_4 = b_4$.

Only voltage differences have meaning $\Rightarrow$ can set $b_2 = 0$.

At ② and ④ there will be current flowing in and out of the battery. Call these z_2 and z_4.
At all other nodes current is still 0. (i.e. $z_i = 0$, $i = 1, 3, 5$)

Equations of circuit:

unknown

$$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \boxed{b_1} \\ b_2 \\ \boxed{b_3} \\ b_4 \\ \boxed{b_5} \end{bmatrix} = \begin{bmatrix} 0 \\ z_2 \\ 0 \\ z_4 \\ 0 \end{bmatrix}$$

known

First rearrange matrix:

$$\begin{array}{c|c} & \begin{array}{ccccc} ② & ④ & ① & ③ & ⑤ \end{array} \\ \begin{array}{c} ② \\ ④ \\ ① \\ ③ \\ ⑤ \end{array} & \left[\begin{array}{cc|ccc} 2 & 0 & -1 & -1 & 0 \\ 0 & 3 & -1 & -1 & -1 \\ \hline -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{array}\right] \end{array} \left[\begin{array}{c} b_2 \\ b_4 \\ \hline b_1 \\ b_3 \\ b_5 \end{array}\right] = \left[\begin{array}{c} z_2 \\ z_4 \\ \hline 0 \\ 0 \\ 0 \end{array}\right].$$

Block matrix form

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \underline{b} \\ \underline{b}' \end{bmatrix} = \begin{bmatrix} \underline{z} \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$$\underbrace{b = \begin{bmatrix} b_2 \\ b_4 \end{bmatrix}, \quad b' = \begin{bmatrix} b_1 \\ b_3 \\ b_5 \end{bmatrix}, \quad z = \begin{bmatrix} z_2 \\ z_4 \end{bmatrix}.}$$ → unknowns!

Our system is:
$$Ab + B^T b' = z$$
$$Bb + Cb' = 0$$

Solve for b': $\boxed{b' = -C^{-1} B b}$

Plug into first equation: $Ab - B^T C^{-1} B b = z$

$$\Rightarrow \boxed{z = (A - B^T C^{-1} B) b}$$

Plug in $b = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

The matrix $\boxed{A - B^T C^{-1} B}$ is the voltage to current map between nodes ② and ④

In our case $A - B^T C^{-1} B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} C^{-1} \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}$

$$= \boxed{1.1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}.$$

$$\Rightarrow b' = -C^{-1} B b = \begin{bmatrix} 3 \\ 2.4 \\ 1.2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_3 \\ v_5 \end{bmatrix}$$

and $z = \begin{bmatrix} z_2 \\ z_4 \end{bmatrix} = \begin{bmatrix} 6.6 \\ -6.6 \end{bmatrix}$.

The "Laplacian" if we isolated ② and ④ from the graph.

In fact, $A - B^T C^{-1} B = \frac{1}{R} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, where R is the reduced resistance between ② and ④.

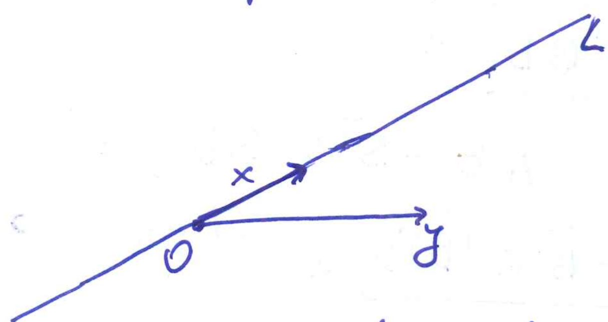
So, if we had R_i any numbers, then $A - B^T C^{-1} B$ would tell us what the resistance between ② and ④ is.

Chapter 3. Orthogonality

3.1. Projections

Projecting onto lines and planes in $\mathbb{R}^3$:

Given a line $L = \text{span}\{x\}$, where $x \in \mathbb{R}^3$ is a fixed (direction) vector, and a point $y \in \mathbb{R}^3$.



Definition: The *projection* of $y \in \mathbb{R}^3$ onto the line $L = \text{span}\{x\}$ is the vector in L that is closest to y. We denote this vector by $P_x y$ or Py (when it is clear what x is).

Q: Given x and y, how do we compute $P_x y$?

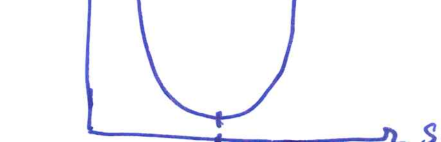
* $P_x y \in L = \text{span}\{x\}$
  $\Leftrightarrow P_x y = s^* x$ for some $s^* \in \mathbb{R}$
* $d^2 = \|P_x y - y\|_2^2 = \|s^* x - y\|^2$ is smallest for s^* among all $s \in \mathbb{R}$!

Let $f(s) = \|sx - y\|^2$. Thus, s^* *minimizes* $f(s)$.

To minimize $f(s)$, need $f'(s) = 0$ ...

But $f(s) = \|sx - y\|^2 = \langle sx - y, sx - y \rangle = \langle sx, sx \rangle - \langle y, sx \rangle - \langle sx, y \rangle + \langle y, y \rangle = s^2 \langle x, x \rangle - 2s\langle x, y \rangle + \langle y, y \rangle$

$= s^2 \|x\|^2 - 2s\langle x, y \rangle + \|y\|^2$ (a parabola looking up / convex)



$$f'(s) = 2\|x\|^2 s - 2\langle x,y\rangle = 0$$

$$\Rightarrow \boxed{s^* = \frac{\langle x,y\rangle}{\|x\|^2}} \quad (\text{assuming } x \neq \vec{0}).$$

$$\Rightarrow \boxed{P_x y = s^* x = \frac{\langle x,y\rangle}{\|x\|^2}\cdot x.}$$

Two ~~interpretations~~ ways to write:

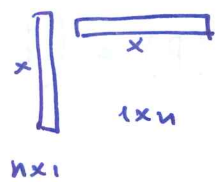
① $$P_x y = \left\langle \frac{x}{\|x\|}, y \right\rangle \underbrace{\frac{x}{\|x\|}}_{=\hat{x}} = \langle \hat{x}, y\rangle \hat{x}$$

$= \hat{x} \rightarrow$ unit vector in the direction of x.

②. Recall that $\langle x,y\rangle = x^T y$.

Then $$P_x y = \frac{1}{\|x\|^2}(x^T y)\,x = \frac{1}{\|x\|^2}\Big(\underset{n\times 1}{x}\ \underset{1\times 1}{(x^T y)}\Big) = \underbrace{\frac{1}{\|x\|^2}}_{\text{scalar}} \underbrace{(x x^T)}_{n\times n \text{ matrix}} y$$

a number



$$\Rightarrow P_x y = \underbrace{\left(\frac{x x^T}{\|x\|^2}\right)}_{P_x} y$$

$P_x \leftarrow$ the matrix associated with the projection operator P_x.

<u>Example</u>: Let $x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Then,

$$P_x = \frac{1}{\|x\|^2} x x^T = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, say $y = \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix}$, we have $$P_x y = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 18/5 \\ 36/5 \\ 0 \end{bmatrix}.$$

(often, we denote P_x by just P).