① ~~5~~

Using new notation:

$$Dv = \begin{bmatrix} j_1 R_1 \\ j_2 R_2 \\ \vdots \\ j_5 R_5 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 & & 0 \\ & R_2 & \\ & & \ddots \\ 0 & & R_5 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_5 \end{bmatrix}}_{\vec{j}}$$

(matrix)

So,

$$Dv = R\vec{j} \Rightarrow R^{-1} D v = \vec{j}$$

where

$$R^{-1} = \begin{bmatrix} 1/R_1 & & 0 \\ & \ddots & \\ 0 & & 1/R_5 \end{bmatrix}.$$

Next, let's calculate the current accumulating at each vertex:

$$\vec{z} = D^T \vec{j} = \left(D^T R^{-1} D\right) v$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

current accumulating at vertex i.

<u>Definition</u>: The matrix

$$L = D^T R^{-1} D$$

is called the Laplacian of the circuit.

<u>Remark</u>: L is symmetric: $L^T = L$.

<u>Pf</u>: $\underbrace{(D^T R^{-1} D)^T}_{L^T} = D^T (R^{-1})^T (D^T)^T$

$$= D^T R^{-1} D = L$$

Ex: Let's calculate the Laplacian of the circuit above, with $R_1 = R_2 = \dots = R_5 = 1$.

$$L = D^T \cdot I \cdot D = D^T D = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Note: If all the resistors are 1Ω, then L is such that:

L_{ii} = # of neighbours of vertex i

$$L_{ij} = \begin{cases} -1 & \text{if vertex } i \,\&\, j \text{ are connected} \\ 0 & \text{if " " not connected} \end{cases}$$

What if $R_j \neq 1$?

$$L_{ii} = \sum_j \frac{1}{R_j}$$, sum over all edges connecting to vertex i

$$L_{ij} = \begin{cases} -\frac{1}{R_k} & (i) \overset{R_k}{\text{—}\!\text{/\textbackslash/}\!\text{—}} (j) \\ 0 & \text{otherwise} \end{cases}$$

Back to currents:

$$\vec{z} = D^T R^{-1} D v = L v$$

(current accumulating @ each node)

Kirchoff's law:

$$L\vec{v} = \vec{0}$$

(no current accumulates at any node (unless there are power sources involved)).

An immediate implication:

For a connected circuit with no power source,

$$L\vec{v} = 0 \iff D^T R^{-1} D v = \vec{0}$$

$$\iff D v = 0 \iff v = c \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

needs proof (*)

In other words, in a connected circuit with no power source, all nodes are at the same voltage level, and no current is flowing through the circuit. for ✱

<u>Prop</u>: $N(L) = N(D)$.

- Let $x \in N(D)$. Then $Dx = \vec{0}$

$\Rightarrow Lx = D^T R^{-1}(Dx) = D^T R^{-1} \vec{0} = \vec{0}$.

$\Rightarrow x \in N(L)$. Then $N(D) \subseteq N(L)$.

- Let $y \in N(L)$. Then

$Ly = D^T R^{-1} D y = \vec{0}$

$\Rightarrow y^T L y = \underbrace{y^T D^T}_{w^T} R^{-1} \underbrace{Dy}_{w} = 0$.

$\Rightarrow w^T R^{-1} w = 0$

$\Leftrightarrow [w_1\ w_2 \cdots w_n] \begin{bmatrix} 1/R_1 & & 0 \\ & \ddots & \\ 0 & & 1/R_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = 0$

$\Leftrightarrow [w_1\ w_2 \cdots w_n] \begin{bmatrix} 1/R_1\, w_1 \\ \vdots \\ 1/R_n\, w_n \end{bmatrix} = 0$.

$\Leftrightarrow \frac{1}{R_1} w_1^2 + \frac{1}{R_2} w_2^2 + \cdots + \frac{1}{R_n} w_n^2 = 0$

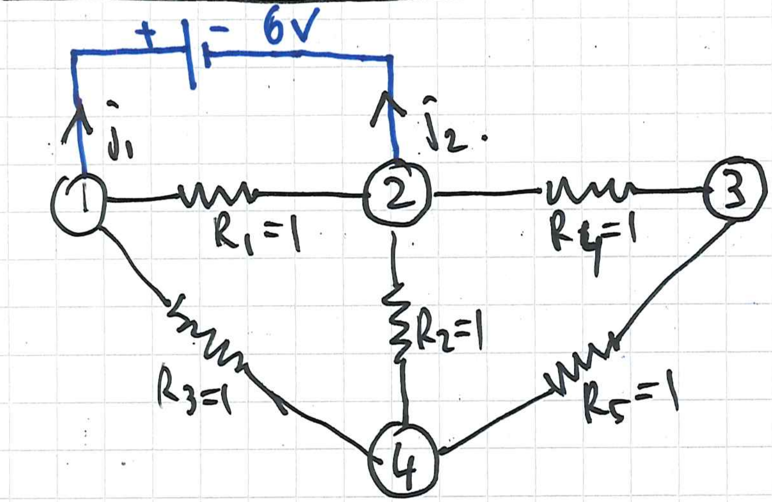
Since $\frac{1}{R_j} > 0$ and $w_j^2 \geq 0$

$\Leftrightarrow w_1 = w_2 = \cdots = w_n = 0$.

$\Leftrightarrow Dy = \vec{0}$ (remember $w = Dy$)

$\Rightarrow y \in N(D)$.

Connecting a battery:



What changes:

$\cancel{v_1 = 6 ; v_2 =}$

- $v_1 = b_1$; $v_2 = b_2$
  (in our example: $v_1 = 6$; $v_2 = 0$).
- j_1 and j_2 are currents such that
  $j_1 = -j_2$.

Let's compute the currents accumulating at each vertex:

$$Lv = z$$

Where:

$$\left[\begin{array}{cc|cc} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ \hline 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{array}\right] \left[\begin{array}{c} v_1 = b_1 \\ v_2 = b_2 \\ \hline v_3 \\ v_4 \end{array}\right] = \left[\begin{array}{c} J_1 \\ J_2 \\ \hline 0 \\ 0 \end{array}\right]$$

Above: b_1, b_2 fixed (known)

v_3, v_4, J_1, J_2 unknown

This system is of the form

unknowns

$$\left[\begin{array}{c|c} A & B^T \\ \hline B & C \end{array}\right] \left[\begin{array}{c} \vec{b} \\ \hline \vec{v} \end{array}\right] = \left[\begin{array}{c} \vec{J} \\ \hline \vec{0} \end{array}\right]$$

where $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$,

...

Then:
$$\boxed{\begin{aligned} Ab + B^T v &= J \quad (1) \\ Bb + Cv &= 0 \quad (2) \end{aligned}}$$
$\longrightarrow$

From (2): $-Bb = Cv$

$$\Rightarrow \boxed{v = -C^{-1}Bb}$$

$\hookrightarrow$ plug into (1):

$$Ab - B^T C^{-1} B b = J$$

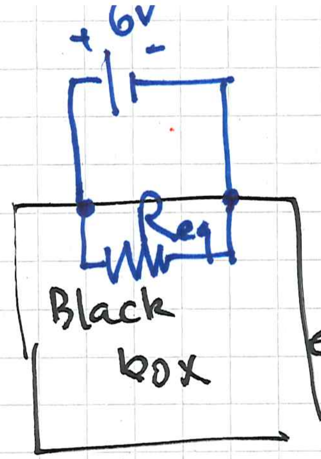
$$\Rightarrow \underbrace{(A - B^T C^{-1} B)}_{M \leftarrow \text{known quantity}} b = J$$

Ex: In our example,

$$M = \frac{8}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 6 \\ 0 \end{bmatrix};$$

$$J = \frac{8}{5}\begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

How to interpret?



$\leftarrow$ our circuit (without the power source)

Req: equivalent resistance for the whole black box.

Current: $\frac{8}{5} \times 6$.

Voltage: 6

$$\Rightarrow R_{eq} = \frac{6}{8/5 \times 6} = \frac{1}{8/5} = 5/8.$$

$\uparrow$ equivalent resistance of the network.