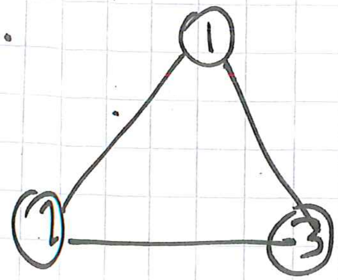
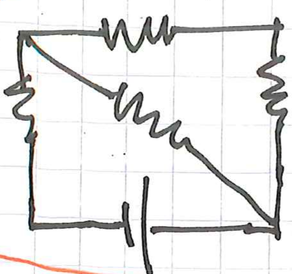
Graphs & Networks

Ex.

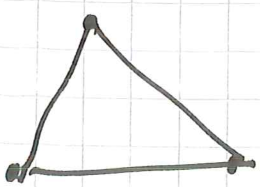


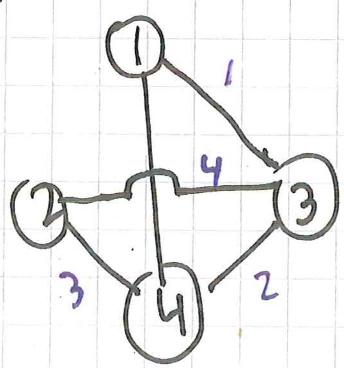
social network

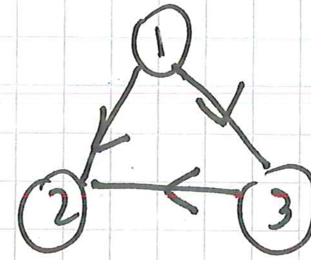


resistor network

A graph is a set of vertices (nodes) and edges.



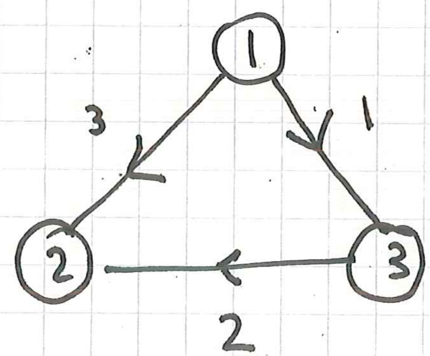




directed graph

We can describe directed graphs (uniquely) via their incidence matrix D

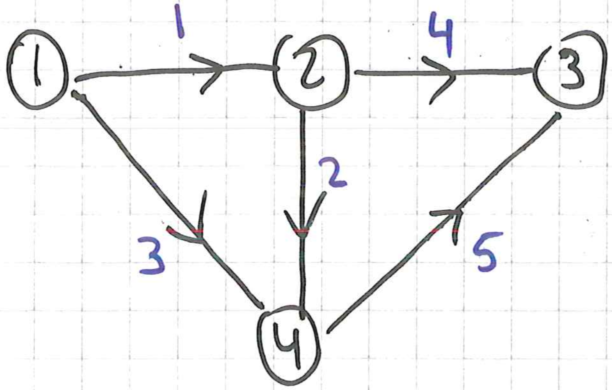
Ex: Consider



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} ① & ② & ③ \end{array} \\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array} = D$$

Incidence matrix of this graph

Ex:



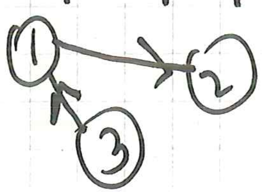
The incidence matrix

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Ex: Given

$$D = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

The corresp. graph is:

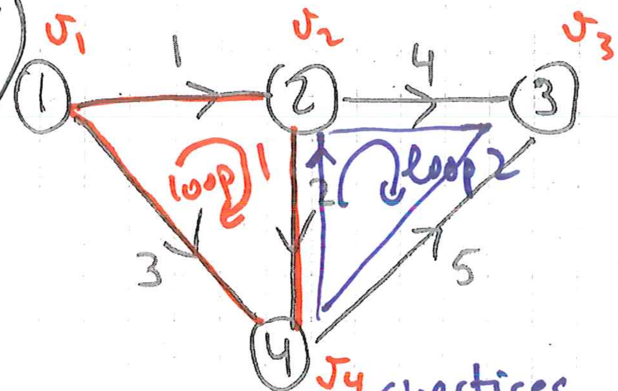


Our next job: understand

$N(D)$, $N(D^T)$

$R(D)$, $R(D^T)$

Original Ex:



vertices

The incidence matrix

$$D = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} ① & ② & ③ & ④ & \\ \end{array}$$

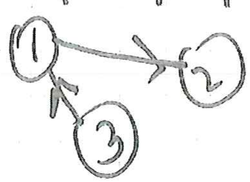
$$\begin{array}{c|cccc|c}
 & 1 & 2 & 3 & 4 & \\
\hline
D = & -1 & 1 & 0 & 0 & 1 \\
 & 0 & -1 & 0 & 1 & 2 \\
 & -1 & 0 & 0 & 1 & 3 \\
 & 0 & -1 & 1 & 0 & 4 \\
 & 0 & 0 & 1 & -1 & 5
\end{array}$$

edges

Ex: Given

$$D = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

The corresp. graph is:



Our next job: Understand

$N(D)$, $N(D^T)$

$R(D)$, $R(D^T)$

this example

$N(D)$: Find the set of all ~~$v \in \mathbb{R}$~~ $v \in \mathbb{R}^4$ (in general $\mathbb{R}^{\#\text{vertices}}$) s.t.

$$Dv = 0.$$

v_j: "voltage" at node j.

$$\Leftrightarrow D\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} v_2 - v_1 \\ v_4 - v_2 \\ v_4 - v_1 \\ v_3 - v_2 \\ v_3 - v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Say $v_1 = s$;

$\Rightarrow v_2 = s$

$\Rightarrow v_4 = s$

$\Rightarrow v_3 = s$

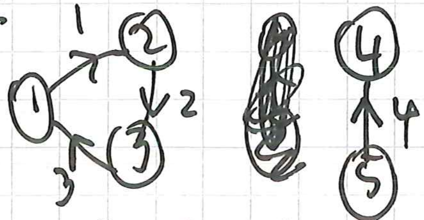
$$\Rightarrow N(D) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Note: If $v \in N(D)$, then the voltages at any nodes that are connected (eventually) have to be identical.

So, ~~for~~ for any connected graph,

$N(D) = \text{span}\left\{ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$ is 1-dimensional.

Ex:



$$D = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$v_1 = v_2 = v_3 = s$

$v_4 = v_5 = t$

$$N(D) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So, $\dim(N(D)) = \#$ connected components of the graph.

② $\underline{R(D)}$: $\{ Dv : v \in \mathbb{R}^{\#\text{vertices}} \}$.

Let v_j: voltage at vertex j.

In our ~~example~~ original example

~~Dv=~~

$$Dv = \begin{bmatrix} v_2 - v_1 \\ v_4 - v_2 \\ \vdots \\ v_3 - v_4 \end{bmatrix}$$

#vertices

$\downarrow$

Note: $\dim(R(D)) = 4 - \dim(N(D))$.

When the graph is connected,

$\dim(N(D)) = 1$

$\Rightarrow \dim(R(D)) = 4 - 1 = 3$

(#vertices − 1)

Let's investigate D^T. Using our

original example:

$$D^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

Consider:

$$D^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

total current accumulating at node #1.

x_i: current along edge i

y_i: current accumulating at vertex # i.

Then: $N(D^T) = \{ z : D^T z = 0 \}$.

no accumulation of current at any node!

In our example,

$$N(D^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

$$D^T \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\text{Col\#1} + \text{Col\#2} - \text{Col\#3}}_{\text{corresponds to loop 1}} \ (\text{of } D^T)$$

edge 1   edge 2   - edge 3

$$D^T \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \underbrace{-\text{Col\#2} + \text{Col\#4} - \text{Col\#5}}_{\text{corresp. to loop 2}}$$

Digression:

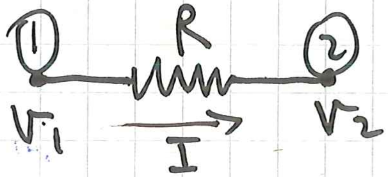
$$Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Let's talk about $\mathcal{R}(D^T)$ later.

Resistor networks (circuits)

Ohm's law :



I: current from ① to ②

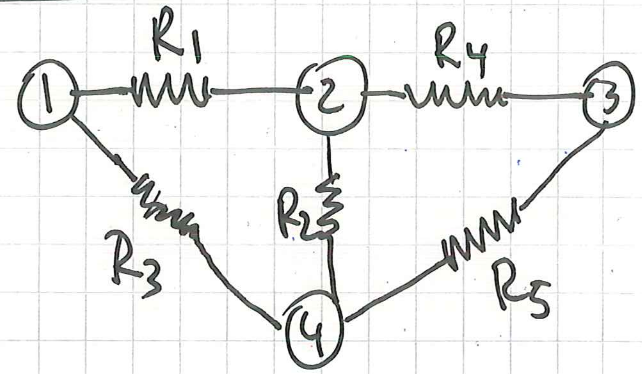
R: resistance (in Ωs)

v_i: Voltage at node i.

$$v_2 - v_1 = I \cdot R \quad \leftarrow \text{Ohm's law}$$

Let's modify our original

example:



As before: v_i: voltage at vertex i.

Let $\vec{j}$ be the vector of currents, i.e., j_i is the current on edge i (across R_i).

For example:

$$v_2 - v_1 = j_1 \cdot R_1$$

$$v_4 - v_2 = j_2 \cdot R_2$$

$$\vdots$$

Using new notation:

$$D v = \begin{bmatrix} j_1 R_1 \\ j_2 R_2 \\ \vdots \\ j_5 R_5 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 & & 0 \\ & R_2 & \\ & & \ddots \\ 0 & & R_5 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_5 \end{bmatrix}}_{\vec{j}}$$

(matrix)

So,

$$D v = R \vec{j} \Rightarrow R^{-1} D v = \vec{j}$$

where $R^{-1} = \begin{bmatrix} 1/R_1 & & 0 \\ & \ddots & \\ 0 & & 1/R_5 \end{bmatrix}$.