Recap: The 4 fundamental spaces of a matrix

$\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A^T)$.

For the following matrix, find those, spaces ~~and~~ their dimensions, and a basis for each one.

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & -1 \\ 1 & -2 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & -2 & 1 & 1 & -6 \end{bmatrix}; \quad \text{rref}(A) = \begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & -4 \\ 0 & 0 & 0 & \boxed{1} & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\left.\begin{aligned} \dim(\mathcal{R}(A)) &= \#\text{pivots} = 3 \\ \dim(\mathcal{R}(A^T)) &= 3 \end{aligned}\right\} = \text{rank}(A) = \text{rank}(A^T).$$

$$\dim(\mathcal{N}(A)) = 5 - \#\text{ pivots} = 2$$

$$\dim(\mathcal{N}(A^T)) = 4 - \#\text{ pivots} = 1.$$

Basis for $\mathcal{R}(A)$: pivot columns: 1, 3, 4.

pick those columns from A: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Basis for $\mathcal{N}(A)$: Solve $\text{rref}(A)x = \vec{0}$:

$\left.\begin{aligned} x_2 &= s \\ x_5 &= t \end{aligned}\right\}$ free variables. Then, $x_1 = 2s - 3t$

$x_3 = 4t$, $x_4 = 5t$.

$$\Rightarrow \mathcal{N}(A) = \left\{ s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \\ 5 \\ 1 \end{bmatrix} \right\}. \quad \text{Basis: } \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

Basis for $\mathcal{R}(A^T)$: pivot rows of rref(A)

same as first #pivots columns in $(\text{rref}(A))^T$:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}.$$

Basis for $N(A^T)$: Easiest would be to find $rref(A^T)$:

$$rref(A^T) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solve $rref(A^T)x = \vec{0}$.

$x_4 = s$ free variable.

$x_1 = -2s$, $x_2 = s$, $x_3 = 2s$.

$$\Rightarrow N(A^T) = \left\{ s \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \quad \text{Basis: } \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Lecture 13: Orthogonal vectors & subspaces [II.2.7 & II.2.8]

Definition: The inner product (~~usual~~ the dot product) of two vectors $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ and $y = [y_1, \dots, y_n]^T \in \mathbb{R}^n$ is

$$\boxed{\langle x, y \rangle := \sum_{j=1}^{n} x_j y_j .}$$

Useful properties:

① We can use matrix notation and write:

$$\langle x, y \rangle = x^T y .$$

② $\langle x, y \rangle = \langle y, x \rangle$

③. $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle \quad \forall a, b \in \mathbb{R},\ x, y, z \in \mathbb{R}^n$.

④. $\langle x, Ay \rangle = \langle A^T x, y \rangle$

Proof: $\langle x, Ay \rangle = x^T A y = (x^T A) y = \langle A^T x, y \rangle$

Note: say $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$

Then $Ay \in \mathbb{R}^n$, $A^T x \in \mathbb{R}^m$.

⑤ $\langle x, x \rangle = \sum_{j=1}^{n} |x_j|^2 = \|x\|_2^2$.

↳ the inner product "induces" the 2-norm.

⑥. $|\langle x, y \rangle| \leq \|x\|_2 \cdot \|y\|_2$

↳ Cauchy-Schwarz inequality.

⑦. $\langle x, y \rangle = \|x\| \cdot \|y\| \cos \theta$

θ ↳ angle between x and y.

Example: what is the angle between $x = [1, 2, 3, 4]^T$ and $y = [-1, 1, 2, -2]^T$?

Sol. $\cos \theta = \frac{\langle x, y \rangle}{\|\dots\|} = \frac{-1}{\dots} = -\frac{1}{10\sqrt{\dots}} \Rightarrow \theta = \arccos\left(\frac{-1}{10\sqrt{3}}\right)$

Orthogonality: x and y are <u>orthogonal</u> if the angle b/w them is $\frac{\pi}{2}$.

Alternatively: x and y are <u>orthogonal</u> iff $\langle x, y\rangle = 0$.

<u>Notation</u>: $x \perp y$.

Examples: (1). $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

Then, $\langle x, y\rangle = -1-1+2 = 0$.

$\Rightarrow$ x and y are orthogonal.

(2). Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$:

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th position}$$

Then, $\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} =: \delta_{ij}$.

Definition: Let S_1 and S_2 be two subspaces. Then, S_1 and S_2 are <u>orthogonal</u> iff

$\forall u \in S_1$, $\forall v \in S_2$, $\langle u, v\rangle = 0$.

In this case, $S_1 \perp S_2$.

<u>How do we decide if two subspaces are orthogonal?</u>

Theorem: Two subspaces S_1 and S_2 ~~...~~ $\subseteq V$ are orthogonal iff their basis vectors are orthogonal. That is,

if ~~...~~ $\{b_1, b_2, \dots, b_k\}$ is a basis for S_1, and

$\{c_1, c_2, \dots, c_\ell\}$ is a basis for S_2,

then $S_1 \perp S_2$ iff $\langle b_i, c_j\rangle = 0$ $\forall i, j$.

Remark: To check this condition:

- let $B := [b_1 | b_2 | \cdots | b_k] \in \mathbb{R}^{m \times k}$, $C := [c_1 | c_2 | \cdots | c_\ell] \in \mathbb{R}^{m \times \ell}$.

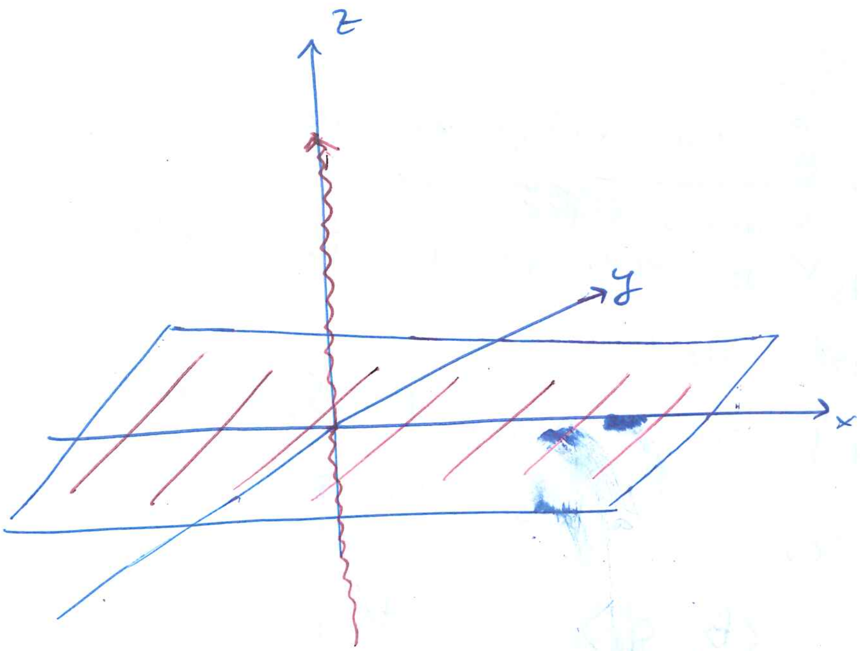
(assuming $S_1, S_2 \subseteq \mathbb{R}^m$)

(Q: Are k and l larger/smaller than m?)

Then, $B^T C = \begin{bmatrix} - b_1^T - \\ - b_2^T - \\ \vdots \\ - b_k^T - \end{bmatrix} [c_1 | c_2 | \cdots | c_\ell]$

$$= \begin{bmatrix} b_1^T c_1 & b_1^T c_2 & \cdots & b_1^T c_\ell \\ b_2^T c_1 & & & \vdots \\ \vdots & & & b_k^T c_\ell \\ b_k^T c_1 & \cdots & \cdots & \end{bmatrix} \in \mathbb{R}^{k \times \ell}$$

Thus, $S_1 \perp S_2 \iff B^T C = 0$ ← all-zero $k \times \ell$ matrix



$S_1 =$ x-y-plane $= \text{span}\{e_1, e_2\}$

$S_2 =$ z axis $= \text{span}\{e_3\}$.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B^T C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow S_1 \perp S_2.$$

Definition: let $\mathcal{U}$ be a subspace of a vector space W.

Define $\mathcal{U}^\perp := \{w \in W : w \perp \mathcal{U}\}$.

(Here $w \perp \mathcal{U} \Leftrightarrow \langle w, u \rangle = 0 \ \forall u \in \mathcal{U}$.)

$\mathcal{U}^\perp$ is called the <u>orthogonal complement</u> of $\mathcal{U}$ in W.

<u>Note</u> that $\mathcal{U}^\perp$ is a subspace!

<u>Proof</u>: If $w, v \in \mathcal{U}^\perp$ and a, b scalars, then

$$\langle aw + bv, u \rangle = a\langle w, u \rangle + b\langle v, u \rangle = 0 \quad \forall u \in \mathcal{U}.$$

Thus, $aw + bv \in \mathcal{U}^\perp$. $\Rightarrow$ It is a subspace.

<u>Examples</u>: (1). $S_1^\perp = S_2$ in example above.

(2). $\mathcal{U} = \text{span}\{e_1, e_3\} \subseteq W = \mathbb{R}^5$

What is $\mathcal{U}^\perp$?

<u>Sol</u>: $\mathcal{U}^\perp = \text{span}\{e_2, e_4, e_5\}$.

<u>Remarks</u>: (1). Given $\mathcal{U}$ and $\mathcal{U}^\perp$ in W, any $x \in W$ can be written (uniquely) as $x = x_u + x_{u^\perp}$, where $x_u \in \mathcal{U}$ and $x_{u^\perp} \in \mathcal{U}^\perp$. (we will see this in chapter III).

(2) $(\mathcal{U}^\perp)^\perp = \mathcal{U}$ (also next chapter)

(3) If $\mathcal{U} \subseteq W$, $\dim(W) = n$, then

$$\dim(\mathcal{U}^\perp) = n - \dim(\mathcal{U}).$$

<u>Orthogonality relations for $R(A), N(A), R(A^T), N(A^T)$:</u>

$$\boxed{\begin{aligned} &(1).\ N(A) = [R(A^T)]^\perp \\ &(2).\ N(A^T) = [R(A)]^\perp. \end{aligned}}$$

Note that (2) follows from (1) by replacing A with A^T (since $(A^T)^T = A$).