10/15/2019

Lecture 12

Last time: • Span, basis, dimension of vector subspaces.

• we started talking about the four fundamental subspaces of a matrix.

$$\mathcal{N}(A),\ \mathcal{R}(A),\ \mathcal{N}(A^T),\ \mathcal{R}(A^T).$$

① $\mathcal{N}(A) = \{x : Ax = \vec{0}\}$.

To find a basis for $\mathcal{N}(A)$:

$\to$ Solve system $Ax = \vec{0}$

$\iff$ Solve $\text{rref}(A)x = \vec{0}$

Example: $C = \begin{bmatrix} 1 & 3 & 3 & 1 & 0 \\ 2 & 6 & -1 & -1 & 0 \\ 1 & 3 & 1 & 4 & 1 \end{bmatrix} \rightsquigarrow \text{rref}(C) = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{0}$$

$\rightsquigarrow x_2 = s,\ x_4 = t$ free variables.

$x_1 = -3s - t$

$x_3 = -3t$

$x_5 = 0$

$$x = s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathcal{N}(C) = \left\{ s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Then $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathcal{N}(C)$!

Fact: This is always the case.

② $\boxed{\mathcal{R}(A) := \{Ax : x \in \mathbb{R}^n\}}$ the range of A / column space of A.

Let $A = [a_1 | a_2 | \cdots | a_n]$, $a_j \in \mathbb{R}^m$

Then, recall that $Ax = x_1a_1 + x_2a_2 + \dots + x_na_n$, a linear combination

Thus, $\boxed{R(A) = \operatorname{span}\{a_1, a_2, \dots, a_n\}}$.

How do we find a basis for $R(A)$?

Equivalently, we need to find a basis for $\operatorname{span}\{a_1, \dots, a_n\}$

Equivalently, we need to find the largest number of linearly independent columns of A.

$\Leftrightarrow$ We need to find the pivot columns of A. Why?

Recall: • Pivot columns of A are linearly independent

$$Ax = \vec{0} \iff \operatorname{rref}(A)x = \vec{0}$$

e.g. $\cancel{[a_1 | a_2 | a_3 | a_4 | a_5]x = \vec{0} \iff \begin{bmatrix} 1 & * & 0 & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \vec{0}}$

$\cancel{x_1 a_1 + x_3 a_3 + x_5 a_5 = \vec{0} \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \vec{0}}$

$$\begin{array}{c} \begin{bmatrix} 1 & 3 & 3 & 10 & 0 \\ 2 & 6 & -1 & -1 & 0 \\ 1 & 3 & 1 & 4 & 1 \end{bmatrix} \\ \begin{matrix} \uparrow & & \uparrow & & \uparrow \\ c_1 & & c_3 & & c_5 \end{matrix} \end{array} x = \vec{0} \iff \begin{bmatrix} 1 & 3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \vec{0}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \iff x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$$

$\iff x_1 = x_2 = x_3 = 0 \Rightarrow$ l.i.

• All other columns can be expressed as linear combinations of the pivot columns:

$$\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \boxed{c_2 = 3c_1}$$

$$\boxed{c_4 = c_1 + 3c_3}$$

Definition: $\dim(R(A))$ is called the rank of A and is denoted by $\operatorname{rank}(A)$.

Facts: (1) $\text{rank}(A) = \#\text{pivots in } \text{rref}(A)$. $R(A) \subseteq \mathbb{R}^m$

~~Let~~ Let A be $m \times n$. Then, (2) $\text{rank}(A) \leq \min(m, n)$. #pivots $\leq$ #cols. $\leq$ #rows

(3) $\underbrace{\dim(N(A))}_{= \#\text{ free variables}} = n - \#\text{ pivots in } \text{rref}(A)$.

$$\Rightarrow \boxed{\dim(N(A)) = n - \text{rank}(A)}$$

The "rank-nullity theorem".

Bases for $N(A^T)$, $R(A^T)$:

Facts: Let A be $m \times n$. Then,

(1) A^T is $n \times m$.

(2). Columns of A^T are rows of A.

(3). ~~Let rref(A) = ... Then,~~ There exists an invertible matrix Q s.t. $\text{rref}(A) = QA$

$$\Leftrightarrow \underbrace{Q^{-1}}_{=:L}\text{rref}(A) = A$$

$\Leftrightarrow A = L\,\text{rref}(A)$ and L is invertible.

$$A^T = (\text{rref}(A))^T L^T$$

Claim: If L is invertible, so is L^T

Exercise: Prove this claim

Theorem: $R(A^T) = R((\text{rref}(A))^T)$

Remark! $(\text{rref}(A))^T \neq \text{rref}(A^T)$!!!

Proof of the Theorem:

$$R(A^T) = \{A^T x : x \in \mathbb{R}^m\} = \{(\text{rref}(A))^T \underbrace{L^T x}_{y \in \mathbb{R}^m} : x \in \mathbb{R}^m\}$$

$$\stackrel{?}{=} \{(\text{rref}(A))^T y : y \in \mathbb{R}^m\}$$

Q: Is it true that $\{L^T x : x \in \mathbb{R}^m\} = \mathbb{R}^m$? I.e., can we find,

$\underline{A}$: yes. In fact given $y \in \mathbb{R}^m$, $x = (L^T)^{-1} y$ and

$y = L^T x$.

Therefore, $R(A^T) = \{(\text{rref}(A))^T y : y \in \mathbb{R}^m\} = R((\cancel{A}\,\text{rref}(A))^T)$.

Example: let C be as before. Find $R(C^T)$.

Solution: $R(C^T) = R((\text{rref}(C))^T)$

$$\text{rref}(C) = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow R(C^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

a basis for C^T.

~~**Example:** let~~

$$\cancel{A = \begin{bmatrix} 1 & 3 & 4 & 5 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 4 & 7 \end{bmatrix}}.$$

~~(a). What is rank(A)?~~

~~(b). dim(N(A))? dim(R(A))?~~

~~dim(N(A^T))? dim(R(A^T))?~~

④. $N(A^T)$.

A^T is $n \times m$

First, note that $\dim(N(A^T)) = m - \dim(R(A^T))$

For finding a basis, it might be easiest to find $\text{rref}(A^T)$.

Recall that $A = L\,\text{rref}(A)$ and $A^T = (\text{rref}(A))^T L^T$.

$$\text{rref}(A) = QA$$

$$(\text{rref}(A))^T = A^T Q^T$$

Note: Last $m - \text{rank}(A)$ columns of $(\text{rref}(A))^T$ are 0

... last $m - \text{rank}(A)$ columns of Q^T are in $N(A^T)$.

They are also independent $\rightarrow$ they are a basis for $N(A^T)$.

Example: Let $A = \begin{bmatrix} 1 & 3 & 4 & 5 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 4 & 7 \end{bmatrix}$

(a) Find $\operatorname{rank}(A)$.

(b) $\dim(N(A)) = ?$ $\quad \dim(R(A)) = ?$

$\dim(N(A^T)) = ?$ $\quad \dim(R(A^T)) = ?$

(c) Find a basis for $N(A)$.

(d) " " " $R(A)$

(e) " " " $R(A^T)$.

Solution: First, calculate $\operatorname{rref}(A)$:

$$A \rightsquigarrow \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 5 & 18 \\ 0 & 0 & 1 & -2 & 12 \end{bmatrix}$$

(a) $\operatorname{rank}(A) = 3$

(b) $\dim N(A) = n - \operatorname{rank}(A) = 5 - 3 = 2$.

$\dim(R(A)) = 3 \qquad \dim(R(A^T)) = 3$

$\dim(N(A^T)) = 3 - \dim(R(A^T)) = 0$.

(c). $\begin{bmatrix} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 5 & 18 \\ 0 & 0 & 1 & -2 & 12 \end{bmatrix}$ $\qquad x_4 = s \quad x_5 = t \qquad x_1 = 2s + 5t$, $\; x_2 = -5s - 18t$, $\; x_3 = 2s - 12t$

$$N(A) = \left\{ s \begin{bmatrix} 2 \\ -5 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -18 \\ -12 \\ 0 \\ 1 \end{bmatrix} \right\}$$

basis

(d). $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \right\}$.

(e). $R(A^T) = R((\operatorname{rref}(A))^T)$

$= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \right\}$.