10/10/2019 Lecture 11

Announcements:

1. Office hours will now take place in my office ESB 4128.
2. Homework 3 and its solutions have been posted. (it is not to be submitted).
3. Midterm will cover everything we learn through Tuesday on Friday, Oct 18 at 6:30PM.

Recap: Last time

→ Vector subspaces: $S \subseteq V$ is a subspace if $\forall u, v \in S$, $\forall a, b \in F$ $au + bv \in S$.

→ linear dependence/independence.

- $v_1, \dots, v_k$ are l.d. if $\exists\, c_j$ not all 0 s.t.

$$\sum_{j=1}^{k} c_j v_j = \vec{0}$$

- If $v_1, \dots, v_k$ are not l.d., then they are l.i.

We showed:

- $v_1, \dots, v_k$ are l.i. $\iff$ $[v_1 | \cdots | v_k] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$ has a unique solution. (i.e. $c = \vec{0}$)

How to check if $[v_1 | \cdots | v_k] c = \vec{0}$ has a unique solution?

- Proposition: For any $x \in \mathbb{R}^k$, $Vx = \vec{0} \iff \mathrm{rref}(V)\, x = \vec{0}$.

- Thus, find $\mathrm{rref}(V)$, for $V = [v_1 | \cdots | v_k]$, and check if it has a unique solution.

Examples: (1) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 13 \end{bmatrix}$

$$V = \begin{bmatrix} 1 & 1 & 7 \\ 2 & 1 & 10 \\ 3 & 1 & 13 \end{bmatrix} \rightsquigarrow \mathrm{rref}(V) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot columns = 2
vectors = 3
⇒ linearly dependent.

Furthermore, $Vx = \vec{0} \iff rref(V)x = \vec{0}$

$\Rightarrow$ can find x by looking at $rref(V)$:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} x = \vec{0} \qquad x = \begin{bmatrix} -3t \\ -4t \\ t \end{bmatrix}. \quad \text{if } t=1,\ x = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow -3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 10 \\ 13 \end{bmatrix} = \vec{0}.$$

Conclusion: ① Pivot columns of V are linearly independent. Other cols of V can be written as lin. combs of pivot cols.
② The corresponding coefficients can be read off from a solution of $rref(V)x = \vec{0}$.

(2). $rref(V) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $V = [v_1 | v_2 | v_3]$.

What can we say?

↳ v_1 and v_2 are l.i.

↳ $rref(V)x = \vec{0} \iff x = \begin{bmatrix} -5t \\ -7t \\ t \end{bmatrix}$

$$\Rightarrow rref(V)\begin{bmatrix} -5 \\ -7 \\ 1 \end{bmatrix} = \vec{0}$$

$$\Rightarrow V\begin{bmatrix} -5 \\ -7 \\ 1 \end{bmatrix} = \vec{0} \qquad -5v_1 - 7v_2 + v_3 = \vec{0}$$

$$\Rightarrow v_3 = 5v_1 + 7v_2.$$

(3). $rref(V) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$, $V = [v_1 | v_2 | v_3]$

$\Rightarrow$ all 3 columns of V are l.i.

$rref(V)x = \vec{0} \iff x = \vec{0}$.

Span of vectors, Basis, and Dimension

Definition: Given a collection of vectors $v_1, v_2, \dots, v_k \in V$, the set

$$S = \left\{ \sum_{j=1}^{k} c_j v_j : c_j \in \mathbb{R} \right\}$$

is the span of $v_1, \dots, v_k$. We write $S = \operatorname{span}\{v_1, \dots, v_k\}$.

Remark: S is the "smallest" subspace of V that contains $v_1, v_2, \dots, v_k$.

Examples: (1). $\operatorname{span}\left\{ \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{e_3} \right\} = \mathbb{R}^3$

(2) $\operatorname{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$ = xy-plane in $\mathbb{R}^3$

(3). $\operatorname{span}\left\{ \begin{bmatrix}1\\0\\2\\4\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\} \to$ a 2-dim plane in $\mathbb{R}^4$.

Definition: A set $\{v_1, \dots, v_k\}$ is a basis for S if

(1). $S = \operatorname{span}\{v_1, \dots, v_k\}$

(2). $v_1, \dots, v_k$ are linearly independent.

Examples above revisited:

(1). $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$

(2). $\left\{ e_1, e_2, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$ is NOT a basis for the xy-plane.

(3). $\left\{ \begin{bmatrix}1\\0\\2\\4\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$ is a basis for $S = \operatorname{span}\left\{ \begin{bmatrix}1\\0\\2\\4\end{bmatrix}; \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$.

on the other hand, it is not a basis for $\mathbb{R}^4$!

Remark: Conditions (1) and (2) in the definition of basis above can be replaced with:

(1)': $\forall u \in S$, $\exists\, c_1, \dots, c_k$ s.t. $u = c_1 v_1 + \dots + c_k v_k$

(2)': The choice of c_j in the above formula is unique!

Exercise: Prove that (1) & (2) are equivalent to (1)' & (2)'.

Important Facts

(1). Given a subspace S, the choice of basis is NOT unique.

Example: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ are both bases for $\mathbb{R}^2$.

(2). Given a subspace S, the number of vectors in any basis for S is the same.

Proof: see the typed notes.

Definition: If $\{v_1, \dots, v_k\}$ is a basis for S, then the dimension of S is k. We write $\dim S = k$.

Theorem: 1. Let $\{u_1, \dots, u_k\}$ be linearly independent. Then, $\{u_1, \dots, u_k\}$ is a basis for its span.

2. Let S be a k-dimensional subspace and suppose $\{v_1, \dots, v_k\} \subseteq S$ is linearly independent. Then, $\{v_1, \dots, v_k\}$ is a basis for S.

Proof: See the typed notes.

Example: Let $S = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ are l.i. $\Rightarrow$ they form a basis for $S \Rightarrow \boxed{\dim S = 2}$.

Also, $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$ are l.i. and lie in $S \Rightarrow \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ is also a basis for S and $S = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

The Four Fundamental Spaces for a Matrix

Remarks: 1. Given an $m \times n$ matrix A, we denote it by

$A = [a_{ij}]_{m \times n}$ and $A^T = [a_{ji}]_{n \times m}$.

$A(ax+by) = aAx + bAy$

2. Recall that a matrix can be interpreted as a linear map (transformation).

Specifically, if A is $m \times n$

$$A: x \in \mathbb{R}^n \longrightarrow y = Ax \in \mathbb{R}^m$$

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

The Four Fundamental Subspaces

1. $\mathcal{N}(A)$: nullspace of A
2. $\mathcal{R}(A)$: range of A
3. $\mathcal{N}(A^T)$
4. $\mathcal{R}(A^T)$.

1. $\mathcal{N}(A)$.

$$\mathcal{N}(A) := \{x \in \mathbb{R}^n : Ax = \vec{0}\} \subseteq \mathbb{R}^n.$$

$\mathcal{N}(A)$ is the solution set to the equation $Ax = \vec{0}$.

Exercise: Show that $\mathcal{N}(A)$ is a subspace.

(If $a, b \in F$ scalars, and $u, v \in \mathcal{N}(A)$

$$A(au + bv) = aAu + bAv = a \cdot \vec{0} + b \cdot \vec{0} = \vec{0})$$

Examples: (a). A is $n \times n$ invertible. What is $\mathcal{N}(A)$?

$$Ax = \vec{0}$$

$$\Leftrightarrow A^{-1}Ax = A^{-1}\vec{0}$$

$$\Leftrightarrow x = \vec{0}. \quad \Rightarrow \mathcal{N}(A) = \{\vec{0}\}$$ "trivial" nullspace.

(b). $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. $\mathcal{N}(B) = ?$

$B x = \vec{0}$ $\quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ $\quad x_3 = t,\ x_1 = 0,\ x_2 = 0$

$\mathcal{N}(B) = \{ t [0 \ldots] : t \in \mathbb{R}\}$ 1-dimensional

In general, to find $\mathcal{N}(A)$, solve the equation

$$Ax = \vec{0}. \iff \text{rref}(A)x = \vec{0}.$$

How do we find a basis for $\mathcal{N}(A)$ for a given A?

↳ Solve the linear system $\text{rref}(A)x = \vec{0}$.

Examples revisited:

(a). A invertible

$$\mathcal{N}(A) = \{\vec{0}\}.$$

e.g. $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b). $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \text{rref}(B)$ already in rref.

$$\mathcal{N}(B) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is a basis for } \mathcal{N}(B).$$

(c). $C = \begin{bmatrix} 1 & 3 & 3 & 10 \\ 2 & 6 & -1 & -1 \\ 1 & 3 & 1 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \underbrace{\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rref}(C)}$

pivots

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

$x_2 = s$ $\quad$ $x_4 = t$

$$\begin{aligned} x_1 &= -3s - t \\ x_2 &= s \\ x_3 &= -3t \\ x_4 &= t \end{aligned} \Rightarrow \mathcal{N}(C) = \left\{ s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Then, $\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}\right\}$ is a basis for $\mathcal{N}(C)$.