Examples: (1) $V = \mathbb{R}^n$, $F = \mathbb{R}$

(2) V = set of all $n \times n$ matrices, $F = \mathbb{R}$ with

$$sM = [sM_{ij}]_{n \times n}$$

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e.g. $f(x) = e^x \sin x$ on $[0,1]$ is a "vector" in V.

(4). V = set of all continuous functions on $\mathbb{R}$.

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- no additive invers of $x = 0.5$ in V
  $-0.5 \notin V$
- not closed under addition:
  $x = 0.6$, $y = 0.8$
  $x + y = 1.4 \notin V$.

Lecture 10

10/08/2019

Vector subspaces

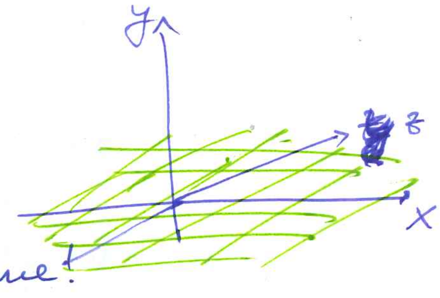
Definition: Let V be a vector space. A subset $S \subseteq V$ is a subspace of V if $\forall u, v \in S$ and $\forall a, b$ scalars:

(1) $u + v \in S$

(2) $au \in S$

$\Leftrightarrow$ (1') $au + bv \in S$

Examples: 1. $S = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\}$ ← xz-plane!



Claim: S is a subspace of $V = \mathbb{R}^3$.

Proof: Let $u = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} \in S$ and let $a, b \in \mathbb{R}$.

Let's check (1') $au + bv = \begin{bmatrix} ax_1 + bx_2 \\ 0 \\ az_1 + bz_2 \end{bmatrix} \in S$ $\Rightarrow$ (1') holds

$\Rightarrow$ S is a subspace of $\mathbb{R}^3$.

Remark: If S is a subspace of V, then S is itself a vector space!

2. $V =$ set of all real-valued functions on $\mathbb{R}$.

$F = \mathbb{R}$

$P_n :=$ set of all polynomials of degree $\leq n$.

Claim: P_n is a subspace of V

Proof: Let $p(x) = a_n x^n + \dots + a_1 x + a_0$

$q(x) = b_n x^n + \dots + b_1 x + b_0$

be two polynomials in P_n.

Let $a, b \in \mathbb{R}$

$$ap(x) + bq(x) = (a \cdot a_n + b \cdot b_n) x^n + \dots + (a \cdot a_1 + b b_1) x + (a a_0 + b b_0)$$

$\in P_n$ as well.

Other important examples / remarks

- $V = \mathbb{R}^n$, $u_j \in \mathbb{R}^n$, $j = 1, \dots, m$

$S = \{ c_1 u_1 + \dots + c_m u_m : c_j \in \mathbb{R} \} = \text{span}\{u_1, \dots, u_m\}$

is also a subspace of V

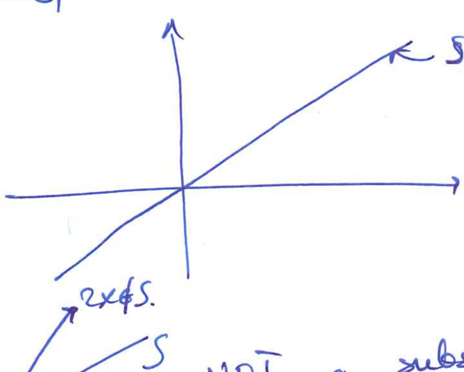
↳ Special case $S = \{c_1 u_1 : c_1 \in \mathbb{R}\}$

line passing through $\vec{0}$ in the direction of u_1!

- In $\mathbb{R}^n$ subspaces are lines, planes, hyperplanes that contain $\vec{0}$.

Example:

(a)



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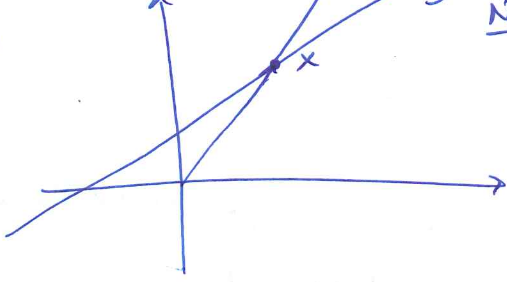

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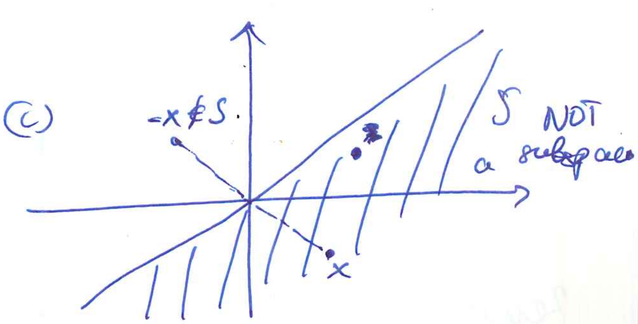

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Lecture 10 10/08/2019

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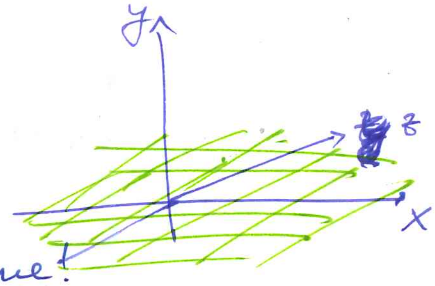

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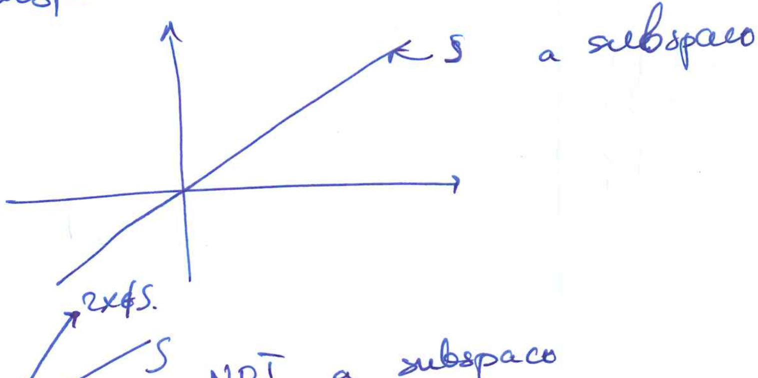

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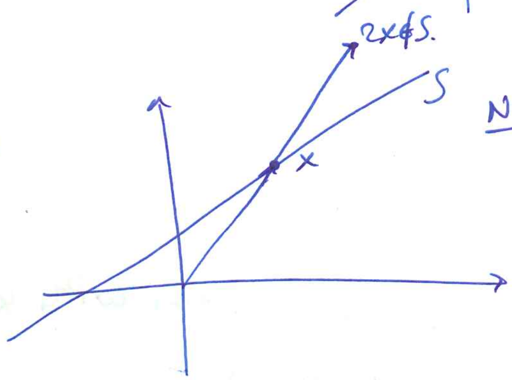
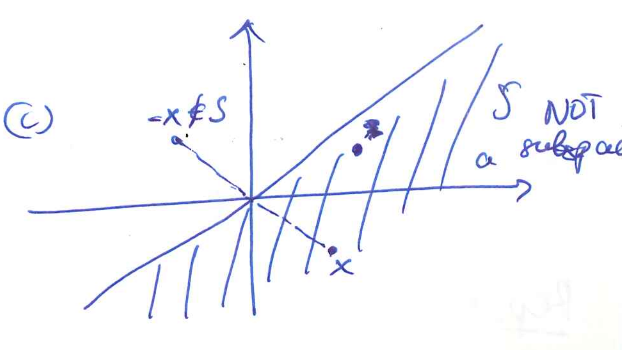


NOT a subspace

(c)



Linear dependence/independence

- Given vectors $\{v_1, \dots, v_k\}$ and scalars $c_1, \dots, c_k \in \mathbb{R}$, the sum
$$c_1 v_1 + \dots + c_k v_k = \sum_{j=1}^{k} c_j v_j$$
is a <u>linear combination</u> of $v_1, \dots, v_k$.

- We say that $\{v_1, \dots, v_k\}$ is <u>linearly ~~in~~dependent</u> if $\exists\, c_j$ <u>not all 0</u> s.t.
$$\sum_{j=1}^{k} c_j v_j = \vec{0}.$$

- If $\{v_1, \dots, v_k\}$ is not linearly dependent, then it is <u>linearly independent</u>.

How to check linear dependence/independence?

<u>Example</u>: (1). $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ is l.d.

$$2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \vec{0}.$$

(2). $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is l.i. <u>because</u>

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array} !$$

(3). $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$ is l.d.

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{0}.$$

(# vectors > dimension → see next class!)

<u>In general</u>: $\{v_1, v_2, \dots, v_k\}$ given.

Are these linearly ~~in~~dependent?

<u>Rephrase</u>: Are there $c_1, \dots, c_k$ not all 0 s.t.

$$\boxed{c_1 v_1 + c_2 v_2 + \dots + c_k v_k = \vec{0}} \; (*) \; ?$$

→ a linear system with k unknowns.

Let's rewrite (*) using matrix notation:

$$\underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}}_{V \quad n\times k} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}}_{c \quad k\times 1} = \underset{n\times 1}{\vec{0}}$$

<u>So</u>: Want to find all solutions to $\boxed{Vc = \vec{0} \; (**)}$

Then,

- if (**) has a unique solution, that solution has to be
  $$c_1 = c_2 = \cdots = c_k = 0.$$
  $\Leftrightarrow \{v_1, \dots, v_k\}$ is linearly independent.
- otherwise, they are linearly dependent

In fact, we can read off this information from the rref of V.

<u>How?</u> <u>Fact from Math 152/221/223/—</u>:

Let A be an $n\times k$ matrix. Then $\exists$ an invertible matrix Q s.t. $QA = \text{rref}(A)$

Alternatively, $A = Q^{-1}\text{rref}(A)$.

Then, we have the following consequence.

$\boxed{\textbf{Proposition: } \text{For any } x \in \mathbb{R}^k, \quad Vx = \vec{0} \Leftrightarrow \text{rref}(V)x = \vec{0}.}$

<u>Proof</u>: Let Q be s.t. $\text{rref}(V) = QV$.

$\Rightarrow$: suppose that $Vx = \vec{0}$. Then, $\underbrace{QV}_{\text{rref}(V)}x = Q\vec{0} = \vec{0}$.

$$\Rightarrow \text{rref}(V)x = \vec{0}.$$

$\Leftarrow$: suppose $\text{rref}(V)x = \vec{0}$

$$\Rightarrow (QV)x = \vec{0}$$
$$\Rightarrow Q(Vx) = \vec{0}$$
$$\Rightarrow \dots Q^{-1}\vec{0} \Rightarrow Vx = 0.$$

Note: above V does not need to be square – works for any $n \times k$ matrix V.

Examples:

(1) $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$ Are these l.d.?

$$V = \begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix} \sim \begin{bmatrix}1&1&1\\0&0&-1\\0&-1&-1\end{bmatrix} \sim \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$\Rightarrow$ unique solution $\Rightarrow$ linearly independent.

(2). $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}7\\10\\13\end{bmatrix} \right\}$

$$V = \begin{bmatrix}1&1&7\\2&1&10\\3&1&13\end{bmatrix} \sim \begin{bmatrix}1&1&7\\0&-1&-4\\0&-2&-8\end{bmatrix} \sim \begin{bmatrix}1&1&7\\0&1&4\\0&2&8\end{bmatrix} \sim \begin{bmatrix}1&0&3\\0&1&4\\0&0&0\end{bmatrix}$$

$\left.\begin{array}{l}\#\text{pivot columns} = 2\\ \#\text{ vectors } = 3\end{array}\right\}$ linearly dependent.

Furthermore, $Vx = \vec{0} \iff \text{rref}(V)x = \vec{0}$

$\Rightarrow$ can find x by looking at rref(V):

col #3 = 3 col #1 + 4 col #2

3 col #1 + 4 col #2 − col #3 = 0

$$\Rightarrow \quad \text{rref}(V)\begin{bmatrix}3\\4\\-1\end{bmatrix} = 0$$

$$\Rightarrow \quad V\begin{bmatrix}3\\4\\-1\end{bmatrix} = 0.$$

Conclusion: (1) Pivot columns of V are linearly independent; other cols can be written as lin. combs of pivot columns
(2). The corresponding coeffs can be read off of rref(V).