Chapter 1: Linear equations.

09/05/2019

Lecture 1

What is a linear equation?

$y = 3x + 2$ ← linear

$y = 3x^2 + 2$ ← NOT linear

$y - 3x - 2 = 0$ ← linear

$x + 2y + xy = 5$ ← NOT linear

$\log(x_1) + x_2 + x_3 = 0$ ← NOT linear.

An equation $f(x_1, \dots, x_n) = b$ is linear iff f is a polynomial of degree 1.

We can always write a linear equation in the form:

$$(*) \quad a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b.$$

Here: x_j are variables

a_j, b are coefficients

We can rewrite $(*)$ using matrix notation

$$[a_1 \; a_2 \; \cdots \; a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [b].$$

A collection of linear equations that have the same solution (thus, we need to solve them simultaneously) are a system of linear equations or a linear system.

Example:
$$\left| \begin{array}{l} 3x_1 + 2x_2 + 3x_3 = 0 \\ x_1 - x_2 + 4x_3 = 1 \end{array} \right. \iff \underbrace{\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix}}_{A:\,2\times 3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b}.$$

Any linear system with m equations and n unknowns can be written as

$$Ax = b$$

A: $m \times n$ matrix, x: $n \times 1$ vector, b: $m \times 1$ vector

How do we solve linear systems?

First question: how many solutions does a linear system have?

$m=1,\ n=1$: $\quad ax=b$

Case 1: $a \neq 0$

$\Rightarrow x = \frac{b}{a}$ $\boxed{\text{unique solution}}$

Case 2: $a=0$

(a) $b \neq 0 \Rightarrow 0 \cdot x = b$ $\boxed{\text{no solution}}$

(b) $b=0 \Rightarrow 0 \cdot x = 0$ every $x \in \mathbb{R}$ is a solution! $\boxed{\text{infinitely many solutions.}}$

$m=2,\ n=2$:

$a_{11}x_1 + a_{12}x_2 = b_1$ ①

$a_{21}x_1 + a_{22}x_2 = b_2$ ②

Let's eliminate x_2:

$a_{22} \times ① - a_{12} \times ②$ gives:

$(a_{22}a_{11} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2 \quad (*)$

Case 1: $a_{22}a_{11} - a_{12}a_{21} \neq 0$.

Then $\boxed{x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}}}$

and plugging in for x_2, we get

$\boxed{x_2 = \frac{-a_{21}b_1 + a_{11}b_2}{a_{22}a_{11} - a_{12}a_{21}}}$. $\boxed{\text{unique solution.}}$

Case 2: $a_{22}a_{11} - a_{12}a_{21} = 0$

(a). $a_{22}b_1 - a_{12}b_2 \neq 0$

$\Rightarrow$ $\boxed{\text{no solution}}$

(b). $a_{22}b_1 - a_{12}b_2 = 0$

$\Rightarrow$ any $x_1 = t \in \mathbb{R}$ will solve $(*)$

Set $x_1 = t$ in ① to get $a_{11}t + a_{12}x_2 = b_1$

$a_{12}x_2 = b_1 - a_{11}t$

$\vdots$

if $a_{12}=0$, $b_1 \neq 0$, $a_{11}=0$: no solution

$\boxed{\text{no solution / infinitely many solutions}}$

In general: Rewrite linear system using matrix notation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then, $\boxed{a_{22}a_{11} - a_{12}a_{21} = \det(A).}$

Thus, it turns out that

Case 1: $\det(A) \neq 0$ $\boxed{\text{unique solution}}$

Case 2: $\det(A) = 0$ $\boxed{\text{no solution / infinitely many solutions}}$

We can define the determinant of any $n \times n$ square matrix (see your notes from your earlier linear algebra class OR our typed notes).

How do we find the solutions to a linear system $Ax = b$?

Gaussian elimination:

- Build augmented matrix $[A \mid b]$
- Using ~~the~~ elementary row operations, bring this augmented matrix into (reduced) row-echelon form.

Example: Solve $\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 1 & -1 & 4 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 5 & -9 & -3 \\ 1 & -1 & 4 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 5 & -9 & -3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 1 & -\frac{9}{5} & -\frac{3}{5} \end{array}\right]$$

pivots

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 11/5 & 2/5 \\ 0 & 1 & -\frac{9}{5} & -3/5 \end{array}\right]$$

pivot columns.

Solution set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ -\frac{3}{5} \\ 0 \end{bmatrix} + t \begin{bmatrix} -11/5 \\ 9/5 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Read off solution set:

row 1: $x_1 = -\frac{11}{5}x_3 + \frac{2}{5}$

row 2: $x_2 = \frac{9}{5}x_3 - \frac{3}{5}$

3 variables, 2 equations

Choose one variable to be "free" $x_3 = t$ (we choose x_3 to be the free variable since column 3 is not a pivot column)

$x_1 = -\frac{11}{5}t + \frac{2}{5}$, $x_2 = \frac{9}{5}t - \frac{3}{5}$.

Example 2:

$$x_1 + 2x_2 = 0$$
$$-3x_1 - 6x_2 = 3$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -3 & -6 & 3 \end{array}\right] \xrightarrow{r2 := r2 + 3r1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 3 \end{array}\right]$$

$$\left| \begin{array}{l} x_1 + 2x_2 = 0 \\ 0.x_1 + 0.x_2 = 3 \end{array} \right. \quad \boxed{\text{No solution!}}$$

Note: $\det \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix} = -6 - (-6) = 0$

Example 3:

$$x_1 - x_2 = 1$$
$$x_1 + x_2 = 3$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{r2 = r2 - r1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 2 \end{array}\right] \xrightarrow{r2 = r2/2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r1 = r1 + r2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \left| \begin{array}{l} x_1 = 2 \\ x_2 = 1 \end{array} \right. \quad \boxed{\text{a unique solution}}$$

Math 307 Section 103 09/05/2019

$\boxed{\text{Lecture 1}}$

1. Go through page.

2. Motivating examples. : where linear algebra is used.
   - ↳ image compression
   - ↳ Google pagerank

Image compression: happens every time we send an image
store it on our hard drive
etc.

it allows us to ~~st~~ reduce the size of the image ~~[illegible]~~ with little perceptible loss in image quality.

100 image 100 pixels $\xrightarrow{\text{FFT}}$ encode image in the frequency domain $\xrightarrow[\text{small frequencies e.g. get rid of 99\% of information}]{\text{truncate}}$ compressed image in frequency domain $\xrightarrow{\text{inverse FFT}}$ 100 compressed image 100 pixels

compressed image in frequency domain: takes up very little space and thus is • stored more efficiently
• ~~sent~~ faster
• etc.

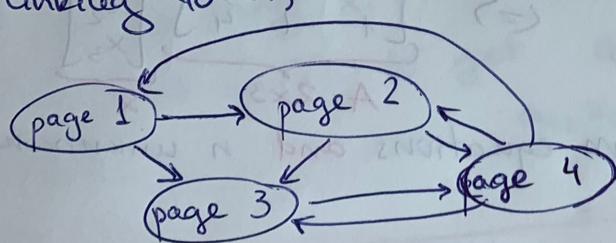
↳ we will learn and implement this.

Google Page Rank Algorithm:

How does Google Search work?

"University of British Columbia"
↳ ubc.ca
↳ wiki page for ubc.ca
Why not a random page containing UBC?

The PageRank Algorithm asseses which pages are most important.
→ the importance of a page is judged by the number of pages linking to it, as well as their importance.



$$H = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & \cdot & \cdot & \frac{1}{3} \\ 2 & \frac{1}{2} & & & \frac{1}{3} \\ 3 & \frac{1}{2} & \frac{1}{2} & & \frac{1}{3} \\ 4 & & \frac{1}{2} & 1 & \end{array}$$

I (page i) = importance of page i

If. $I = \begin{pmatrix} I(\text{page } 1) \\ \vdots \\ I(\text{page } 4) \end{pmatrix}$, then $\boxed{H \cdot I = I}$ I is an eigenvector!!! of H.

$\boxed{\text{25 billion pages}}$

How can we ~~[illegible]~~ compute the page rank?
↳ Power method. to find top eigenvector. We will learn how to implement and show it works.