Math 307, Section 103, Quiz 4

Name: Solutions Student ID#:

Problem 1. (7 points.) Let u and v be two vectors in $\mathbb{C}^2$.

(a) (2 points) Under what conditions do u and v form an orthonormal basis?

$\|u\| = 1$

$\|v\| = 1$

$\langle u, v \rangle = 0$

(b) (2 points) Let $u = \frac{1}{2}\begin{bmatrix} i \\ \sqrt{3} \end{bmatrix}$. Find a vector v so that u and v form an orthonormal basis of $\mathbb{C}^2$.

$\langle u, v \rangle = 0 \iff \underbrace{\bar{i}}_{=-i} v_1 + \sqrt{3} v_2 = 0$

$\iff v_2 = \frac{i v_1}{\sqrt{3}} \Rightarrow v = v_1 \begin{bmatrix} 1 \\ i/\sqrt{3} \end{bmatrix}$

$\boxed{v = \begin{bmatrix} \sqrt{3}/2 \\ i/2 \end{bmatrix}}$.

$\|v\|^2 = 1 \iff |v_1|^2 + |v_1|^2 \left|\frac{i}{\sqrt{3}}\right|^2 = 1$

$\iff |v_1|^2 \left(1 + \frac{1}{3}\right) = 1 \iff |v_1|^2 = \frac{3}{4}$. Let $v_1 = \frac{\sqrt{3}}{2}$

$v_2 = \frac{i}{2}$

$\boxed{\text{Note: any } v_1 = e^{i\theta}\frac{\sqrt{3}}{2} \text{ works!}}$

(c) (3 points) Find the coefficients c_1 and c_2 in the expansion $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 u + c_2 v$.

$c_1 = \left\langle u, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \frac{\bar{i}}{2} = -\frac{i}{2}$

$c_2 = \left\langle v, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \frac{\sqrt{3}}{2}$

Check: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \stackrel{?}{=} -\frac{i}{2}\begin{bmatrix} i/2 \\ \sqrt{3}/2 \end{bmatrix} + \frac{\sqrt{3}}{2}\begin{bmatrix} \sqrt{3}/2 \\ i/2 \end{bmatrix}$

$= \begin{bmatrix} 1/4 \\ -\frac{\sqrt{3}i}{4} \end{bmatrix} + \begin{bmatrix} 3/4 \\ i\sqrt{3}/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ok.

Problem 2. (9 points) Let A be a 2×2 invertible matrix. Suppose that $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$.

(a) (4 points) Show that v_1 is also an eigenvector for A^{-1}.

$$A^{-1}(Av) = A^{-1}(3v)$$

But $A^{-1}Av = v$

Thus, $v = A^{-1}(3v) = 3A^{-1}v$

$$\Leftrightarrow \boxed{A^{-1}v = \tfrac{1}{3}v.}$$

For (b) and (c) let $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be an eigenvector of A with the same eigenvalue $\lambda = 3$.

(b) (3 points) Show that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is also an eigenvector of A with eigenvalue 3.

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} - A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(c) (2 points) Find A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} a = 3 \\ c = 0 \end{matrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Leftrightarrow \begin{matrix} b = 0 \\ d = 3. \end{matrix}$$

Thus, $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$