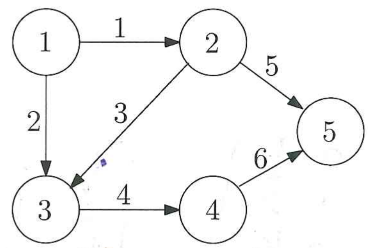
Math 307, Section 103, Quiz 3

Name: Student ID#:

Problem 1. (9 points.) Consider the graph below



(a) (2 points) Write down the incidence matrix D for this graph.

$$D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}.$$

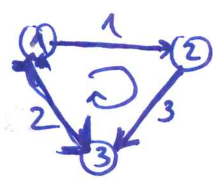
(b) (2 points) What is ~~Find~~ the nullspace $\mathcal{N}(D)$? ~~and~~ Explain what $\dim(\mathcal{N}(D))$ means in terms of the graph.

$$\mathcal{N}(D) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}. \qquad \dim(\mathcal{N}(D)) = 1 = \#\text{ connected components.}$$

(c) (2 points) Find a vector in $\mathcal{N}(D^T)$ and explain what it means in terms of the graph.

Let $v = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $D^T v = 0$.

v is the loop vector for the loop



(d) (2 points) Write down the Laplacian matrix L for this graph, thought of as a resistor network with all resistances equal to 1.

$$L = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix} \end{array}.$$

(e) (1 point) If v_i is the voltage at vertex i, then what is the current Lv for this network?

$Lv = 0$ by Kirchoff's law since there are no power sources connected.

Problem 2. (10 points) Let $P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$.

(a) (4 points) Prove that P is an orthogonal projection matrix.

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P$$

$$P^T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P. \quad \Rightarrow P \text{ is an orthogonal projection matrix.}$$

(b) (1 point) Which subspace S does P project orthogonally to?

P projects onto $\mathcal{R}(P) \left(= \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$.

(c) (3 points) Let $Q = (I - P)$. Find a vector x such that $Qx = x$.

$$Qx = x \iff (I-P)x = x \iff x - Px = x \iff Px = 0.$$

If $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then $Px = 0 \iff Qx = x$.

(d) (2 points) Compute Q^4.

Q is also an orthogonal projection matrix (from class). Thus, $Q^2 = Q$, and, therefore, $Q^n = Q^2 Q^{n-2} = Q^{n-1}, \forall n$

$$\Rightarrow Q^n = Q^{n-1} = \cdots = Q, \quad \forall n$$

Thus, $Q^4 = Q = I - P = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$.